THE K-THEORY OF TOEPLITZ C^*-ALGEBRAS OF RIGHT-ANGLED ARTIN GROUPS

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ABSTRACT. Toeplitz C^*-algebras of right-angled Artin groups were studied by Crisp and Laca. They are a special case of the Toeplitz C^*-algebras $\mathcal{T}(G, P)$ associated with quasi-lattice ordered groups (G, P) introduced by Nica. Crisp and Laca proved that the so called "boundary quotients" $C_Q^*(\Gamma)$ of $C^*(\Gamma)$ are simple and purely infinite. For a certain class of finite graphs Γ we show that $C_Q^*(\Gamma)$ can be represented as a full corner of a crossed product of an appropriate C^*-subalgebra of $C_Q^*(\Gamma)$ built by using $C^*(\Gamma')$, where Γ' is a subgraph of Γ with one less vertex, by the group $\mathbb{Z}$. Using induction on the number of the vertices of Γ we show that $C_Q^*(\Gamma)$ are nuclear and belong to the small bootstrap class. We also use the Pimsner-Voiculescu exact sequence to find their K-theory. Finally we use the Kirchberg-Phillips classification theorem to show that those C^*-algebras are isomorphic to tensor products of $\mathcal{O}_n$ with $1 \le n \le \infty$.

1. INTRODUCTION

Toeplitz C^*-Algebras of right-angled Artin Groups generalize both the Toeplitz algebra and the Cuntz algebras. Coburn showed in [4] that the C^*-algebra, generated by a single nonunitaty isometry is unique, i.e. every two C^*-algebras, each generated by a single nonunitary isometry are *-isomorphic. Similar uniqueness theorems about C^*-algebras generated by isometries were proved by Cuntz [7], Douglas [10], Murphy [13], and others. Laca and Raeburn in [12] and Crisp and Laca in [5] proved such uniqueness theorems for a large class of C^*-algebras, corresponding to quasi-lattice ordered groups (G, P). One of the key point they use was to project onto the "diagonal" C^*-algebra generated by the range projections of those isometries, an idea originating from [10].

These C^*-algebras can be viewed as crossed products of commutative C^*-algebras (the C^*-algebras generated by the range projections of the isometries) by semigroups of endomorphisms. Crisp and Laca used techniques from [11] about such crossed products together with the uniqueness theorems mentioned above to prove a sructure theorem for the universal C^*-algebra $C^*(G, P)$ (which by the uniqueness theorems is isomorphic to the "reduced one" $\mathcal{T}(G, P)$) for a large class of quasi-lattice ordered groups (G, P). We will now state [6, Corollary 8.5] and [6, Theorem 6.7] and use them throuought this note. A graph will always mean a simple graph with countable set of vertices.

Theorem 1.1 ([6], Theorem 6.7). *Suppose that Γ is a graph with a set of vertices S (finite or infinite) such that Γ^{opp} has no isolated vertices. Then the universal C^*-algebra with generators $\{V_s | s \in S\}$ subject to the relations:*
(1) $V_s^* V_s = I$ for each $s \in S$;
(2) $V_s V_t = V_t V_s$ and $V_s^* V_t = V_t V_s^*$ if s and t are adjacent in Γ;
(3) $V_s^* V_t = 0$ if s and t are distinct and not adjacent in Γ;
(4) $\prod_{s \in S_\lambda} (I - V_s V_s^*) = 0$ for each $S_\lambda \subset S$ spanning a finite connected component of Γ^{opp},
is purely infinite and simple.

We will denote the C^*-algebra from this theorem by $C_Q^*(\Gamma)$.

Theorem 1.2 ([6], Corollary 8.5). *Suppose that Γ is a graph with a set of vertices S (finite or infinite) such that Γ^{opp} has no isolated vertices. Let $C^*(\Gamma)$ denote the universal C^*-algebra with generators $\{V_s | s \in S\}$ subject to the relations:*
(1) $V_s^* V_s = I$ for each $s \in S$;
(2) $V_s V_t = V_t V_s$ and $V_s^* V_t = V_t V_s^*$ if s and t are adjacent in Γ;
(3) $V_s^* V_t = 0$ if s and t are distinct and not adjacent in Γ;
Then each quotient of $C^*(\Gamma)$ is obtained by imposing a further collection of relations of the form
(R) $\prod_{s \in S_\lambda} (I - V_s V_s^*) = 0$, where each $S_\lambda \subset S$ spans a finite union of finite connected components of Γ^{opp}.

We remind that by definition the opposite graph of the graph Γ is

$$\Gamma^{\mathrm{opp}} = \{(v, w) | v, w \in S, \ (v, w) \notin \Gamma\}.$$

Γ^{opp} is also called the complement or the inverse of the graph Γ.

Let Γ be a finite graph with set of vertices S such that the opposite graph Γ^{opp} is connected and has more than 1 vertex. Then $C_Q^*(\Gamma)$ is the quotient of $C^*(\Gamma)$ by the ideal generated by $\prod_{s \in S} (I - V_s V_s^*)$. Let $I_\Gamma : \langle \prod_{s \in S} (I - V_s V_s^*) \rangle_{C^*(\Gamma)} \to C^*(\Gamma)$ be the inclusion map of this ideal, and $Q_\Gamma : C^*(\Gamma) \to C_Q^*(\Gamma)$ be the quotient map. Theorem 1.2 implicitly contains the uniqueness theorem ([5, Theorem 24]). In particular we have the following faithful representation $\pi_\Gamma : C^*(\Gamma) \to \mathcal{B}(H_\Gamma)$ which corresponds to $\mathcal{T}(A_\Gamma, A_\Gamma^+)$, where $A_\Gamma = \{S | ss' = s's \text{ if } (s, s') \in \Gamma\}$:
Let H_Γ be the Hilbert space with an orthonormal basis

$$\{\mathfrak{E}[s_1, s_2, \ldots, s_n] | \ n \in \mathbb{N}_0, s_1, \ldots, s_n \in S\} / \sim,$$

where the relation $\sim$ means $\mathfrak{E}[s_1, s_2, \ldots, s_n] \sim \mathfrak{E}[s_1', s_2', \ldots, s_m']$ if and only if $V_{s_1} \cdots V_{s_n} = V_{s_1'} \cdots V_{s_m'}$ subject to commutation relation (2) from Theorem 1.2.
Let π_Γ be given on a generating family of operators and vectors by

$$\pi_\Gamma(V_s)(\mathfrak{E}[\emptyset]) = \mathfrak{E}[s],$$

$$\pi_\Gamma(V_s)(\mathfrak{E}[s_1, s_2, \ldots, s_n]) = \mathfrak{E}[s, s_1, s_2, \ldots, s_n].$$

For this representation it is true that the ideal $\langle \pi_\Gamma(\prod_{s \in S} (I - V_s V_s^*)) \rangle_{\pi_\Gamma(C^*(\Gamma))}$ coinsides with $\mathcal{K}(H_\Gamma)$ - the compact operators on H_Γ.

In [7] Cuntz introduced a certain type of C^*-algebras $\mathcal{O}_n$, $n = 2, 3, \ldots, \infty$ generated by a set of isometries with mutually orthogonal ranges. He was able to represent $\mathcal{K} \otimes \mathcal{O}_n$ as a crossed product of an AF-algebra by $\mathbb{Z}$ ($\mathcal{K}$ stands for the C^*-algebra of the compact operators on a separable Hilbert space). There have been generalizations of these algebras that depend on the "crossed product by $\mathbb{Z}$" idea, for example Cuntz-Krieger algebras [9], Cuntz-Pimsner algebras [17] and others.

In our note for a fixed finite graph with at least three vertices Γ with Γ^{opp} connected we choose a subgraph Γ' one less vertex such that $(\Gamma')^{\mathrm{opp}}$ is connected. Then we represent $C_Q^*(\Gamma)$ as a full corner of a crossed product of a C^*-algebra, built by using $C^*(\Gamma')$, by the group $\mathbb{Z}$. After doing so we can use some results about C^*-algebras which are crossed products by $\mathbb{Z}$. Most importantly we use the Pimsener-Voiculescu exact sequence for the K-thoery ([18]). Using induction on the number of the vertices of the graph we conclude that $C_Q^*(\Gamma)$ is nuclear and belong to the small bootstrap class (see [2, IV.3.1], [1, §22]) and thus the classification result for purely infinite simple C^*-algebras of Kirchberg-Phillips [16] applies. From this we conclude that $C^*(\Gamma)$ is isomorphic to $\mathcal{O}_{1+|\chi(\Gamma)|}$, where $\chi(\Gamma)$ is an analogue of Euler characteristic, introduced in [6]. Then we extend this result to the case when Γ is an infinite graph with countably many vertices and such that Γ^{opp} is connected, since this graph can be represented as an increasing sequence of finite subgraphs. The general case is a graph Γ with at least two and at most countably many vertices which is such that Γ^{opp} has no isolated vertex. It can be treated easily using Theorem 1.1 and the special cases described above. The conclusion is that $C_Q^*(\Gamma)$ is isomorphic to tensor products of $\mathcal{O}_n$ for $1 \leq n \leq \infty$, where we define $\mathcal{O}_1$ to be the unital Kirchberg algebra with $\mathbf{K}_0(\mathcal{O}_1) = \mathbb{Z}[1_{\mathcal{O}_1}]_0$ and $\mathbf{K}_1(\mathcal{O}_1) = \mathbb{Z}$. A Kirchberg algebra is by definition a separable, nuclear, simple, purely infinite C^*-algebra that satisfies the Universal Coefficient Theorem.

2. Some C^*-Subalgebras of $C_Q^*(\Gamma)$ and the Crossed Product Construction

If Γ has two vertices and no edges, then from the construcion of $C^*(\Gamma)$ is clear that $C^*(\Gamma)$ is generated by isometries V_1 and V_2 with orthogonal ranges and such that $V_1 V_1^* + V_2 V_2^* < I$. This is the C^*-algebra $\mathcal{E}_2$ from [8] which is an extension of $\mathcal{O}_2$ by the compacts. Thus $C_Q^*(\Gamma) \cong \mathcal{O}_2$.

Suppose now that Γ has a set of vertices S such that $2 < \mathrm{card}(S) < \infty$ and suppose that Γ^{opp} is connected. Since Γ^{opp} is connected if it is not a tree we can remove an arbitrary edge from its arbitrary cycle and the graph obtained in this way (let's denote it by Γ_1^{opp}) will remain connected. Continuing in this fashion in finitely many (say l) steps we will arrive at Γ_l^{opp} wich will be a tree. Let $s \in S$ be a "leaf" for Γ_l^{opp}. Removing s and the edge that comes out of s from Γ_l^{opp} will not alter the connectedness. All this shows that if Γ' is the graph, obtained from Γ by removing the vertex s and all the edges that come out of s, then its opposite graph $(\Gamma')^{\mathrm{opp}}$ will be connected.

Let $S' \subset S$ be the set of edges of Γ'. We can suppose that $S = \{1, \ldots, n, n+1\}$ and that $S' = \{1, \ldots, n\}$ for some $n \geq 2$. We want to describe the words in letters $\{V_1, \ldots, V_n, V_{n+1}, V_1^*, \ldots, V_n^*, V_{n+1}^*\}$.

Lemma 2.1. *Every word in letters $\{V_1, \ldots, V_n, V_{n+1}, V_1^*, \ldots, V_n^*, V_{n+1}^*\}$ can be written in the form $w_1 w_2^*$, where w_1, w_2 are words in letters $\{V_1, \ldots, V_n, V_{n+1}\}$.*

Proof. We will use induction on the length of the words. The words of length one are V_i and V_i^* and they are of such form. Suppose that the statement of the lemma is true for all words of length $m > 1$ and less. Take a word w of length $m + 1$. We have two cases for w:
1) $w = w' V_i^*$ and 2) $w = w' V_i$ for some $1 \leq i \leq n+1$ and some word w' of length m. By the induction hypothesis w' can be represented as $w' = w_1'(w_2')^*$, where w_1' and w_2' are words in letters $\{V_1, \ldots, V_n, V_{n+1}\}$. In case 1) $w = w_1'(w_2')^* V_i^*$, so setting $w_1 = w_1'$ and $w_2 = V_i w_2'$ shows that w can be written in the desired form. For case 2) if the word w_2' is empty then setting $w_1 = w_1' V_i$ and $w_2 = I$ shows that w has the desired form. If $w_2' = V_j w_2''$ with w_2'' a word in letters $\{V_1, \ldots, V_{n+1}\}$ then

$$w = w_1'(w_2'')^* V_j^* V_i = \begin{cases} 0, & \text{if } (i, j) \notin \Gamma \\ w_1'(w_2'')^*, & \text{if } i = j \\ w_1'(w_2'')^* V_i V_j^*, & \text{if } (i, j) \in \Gamma. \end{cases}$$

The first and the second case in the above equation are words of the desired form. In the third case we have that $w_1'(w_2'')^* V_i$ is a word of length m so it can be represented as $\omega_1 \omega_2^*$. Then $w_1'(w_2'')^* V_i V_j^* = \omega_1 \omega_2^* V_j^*$ is of the desired form. This concludes the induction and proves the lemma. $\qquad\square$

Let's denote by V the isometry $V_{n+1} \in C_Q^*(\Gamma)$ and suppose without loss of generality that $V^* V_i = 0$ for $k < i \leq n$ (notice that since Γ^{opp} is connected, $k < n$). If $k > 0$ then also V commutes and $*$-commutes with $V_1, \ldots, V_k$.

Let $T_0 = C^*(V_1, \ldots, V_n)$. Then from Theorem 1.2 it is easy to see that $T_0 \cong C^*(\Gamma')$. Define by induction T_m to be the closed linear span of elements of $C_Q^*(\Gamma)$ of the form $wV t_{m-1} V^*(w')^*$, where w, w' are words in letters $\{V_1, \ldots, V_n\}$ and $t_{m-1} \in T_{m-1}$. The following lemma characterizes the sets T_m.

Lemma 2.2. *T_m is a C^*-subalgebra of $C^*(\Gamma)$, isomorphic to $\mathcal{K}^{\otimes m} \otimes T_0$ ($\cong \mathcal{K} \otimes C^*(\Gamma')$).*

Proof. Let us denote by Ω the set of all words ω in letters $\{V_1, \ldots, V_n\}$ such that the letters of the word ωV cannot be commuted pass V, i.e. $\omega V = \omega_1 V \omega_2$ for some words ω_1, ω_2 in letters $\{V_1, \ldots, V_n\}$, implies $\omega_2 = I$. It is easy to see that from the connectedness of Γ^{opp} follows that Ω is an infinite countable set therefore we can enumerate its elements: $\Omega = \{\omega_0, \omega_1, \omega_2, \ldots\}$, setting $\omega_0 = I$. We assume that the words in Ω don't repeat, i.e. $\omega_p \neq \omega_q$ for $p \neq q$ after using the commutation relation. Suppose by induction that $T_{m-1} \cong \mathcal{K}^{\otimes(m-1)} \otimes T_0$ for some $m \geq 1$. We want to show that $T_m \cong \mathcal{K} \otimes T_{m-1}$. Clearly $\{\omega_p V t_{m-1} V^* \omega_q^* | p, q \in \mathbb{N}_0\}$ is a $*$-closed set. It is easy to see that each element $w' V t_{m-1} V^* w^*$ ot T_m after applying the commutation relations (2) from Theorem 1.2 can be written in the form $\omega_p V t_{m-1}' V^* \omega_q^*$ for some $p, q \in \mathbb{N}_0$

and some $t'_{m-1} \in T_{m-1}$. Therefore $\{\omega_p V t_{m-1} V^* \omega_q^* | p, q \in \mathbb{N}_0, \ t_{m-1} \in T_{m-1}\}$ spans a dense subset of T_m. We conclude that T_m is $*$-closed.

We want to show now that $V^* \omega_q^* \omega_p V = \delta_{p,q} I$. Write $\omega_p = V_{j_1} \cdots V_{j_s}$ and $\omega_q = V_{i_1} \cdots V_{i_t}$. Then $V^* \omega_q^* \omega_p V = V^* V_{i_t}^* \cdots V_{i_2}^* V_{i_1}^* V_{j_1} V_{j_2} \cdots V_{j_s} V$. There are three cases:

1) If V_{j_1} commutes with $V_{i_1}^*, \ldots, V_{i_r}^*$ ($1 \le r < t$) and $i_{r+1} = j_1$ then $V_{i_{r+1}}^*$ will commute with $V_{i_1}^*, \ldots, V_{i_r}^*$, so the word ω_q can be written in the form $\omega_q = V_{i_1} V_{i_2} \cdots V_{i_t}$ with $i_1 = j_1$. Then we can write $V^* \omega_q^* \omega_p V = V^* V_{i_t}^* \cdots V_{i_2}^* V_{j_2} \cdots V_{j_s} V$ and continue the argument with this word.

2) If V_{j_1} commutes with $V_{i_1}^*, \ldots, V_{i_r}^*$ ($1 \le r < t$) and $(j_1, i_{r+1}) \notin \Gamma$, then $V^* \omega_q^* \omega_p V = 0$. Also if $j_1 > k$ and V_{j_1} commutes with $V_{i_1}^*, \ldots, V_{i_t}^*$ we also have $V^* \omega_q^* \omega_p V = 0$.

3) If V_{j_1} commutes with $V_{i_1}, \ldots, V_{i_t}$ and V clearly then $j_1 \le k$ and from the definition of Ω follows that V_{j_1} doesn't commute with all $V_{j_2}, \ldots, V_{j_s}$. Suppose that V_{j_1} doesn't commute with V_{j_r} ($2 \le r \le s$) and if $r > 2$ V_{j_1} commutes with $V_{j_2}, \ldots, V_{j_{r-1}}$. Notice that $j_r \notin \{i_1, \ldots, i_t\}$ since V_{j_1} commutes with $V_{i_1}^*, \ldots, V_{i_t}^*$ and not with V_{j_r}.

Suppose that $V^* \omega_q^* \omega_p V \ne 0$. Then suppose that $V_{j_1}, \ldots, V_{j_{r_1}}$ can be dealt with by using repeatedly case 1). If $r_1 = s = t$ then $V^* \omega_q^* \omega_p V = \delta_{p,q} I$ is proven. If $r_1 = s < t$ then $V^* \omega_q^* \omega_p V$ reduces to $V^* V_{i_t}^* \cdots V_{i_{s+1}}^* V$. If $i_t \le k$ then $V_{i_t}^*$ would commute with V^* contradicting the fact that $\omega_q \in \Omega$. $i_t > k$ implies immediatelly $V^* V_{i_t}^* \cdots V_{i_{s+1}}^* V = 0$ because V does not commute with all of $V_{i_t}^*, \ldots, V_{i_{s+1}}^*$ so it has a orthogonal range with some of them. The case $r_1 = t < s$ is similar. If $r_1 < s$ and $r_1 < t$ then suppose that for $V_{j_{r_1+1}}$ case 3) applies. We will obtain a contradiction with the fact that $\omega_p \in \Omega$. By case 3) we can find $r_2 > r_1 + 1$ such that $V_{j_{r_1+1}}$ doesn't commute with $V_{j_{r_2}}$ and if $r_2 > r_1 + 2$ then $V_{j_{r_1+1}}$ commutes with $V_{j_{r_1+2}}, \ldots, V_{j_{r_2-1}}$. Also $j_{r_2} \notin \{i_{r_1+1}, \ldots, i_t\}$ ($V_{j_{r_1+1}}$ commutes with $V_{i_{r_1+1}}^*, \ldots, V_{i_t}^*$ and not with $V_{j_{r_2}}$) and so case 1) cannot be applied to $V_{j_{r_2}}$. We can repeat this process finitely many times until we reach the isometry V_{j_s} for which case 3) must apply since case 1) cannot be applied as we saw above and case 2) cannot be applied by assumption. But then $j_s \le k$ and V_{j_s} commutes with V which contradicts $\omega_p \in \Omega$. This proves $V^* \omega_q^* \omega_p V = \delta_{p,q} I$.

It follows that $\omega_p V t_{m-1} V^* \omega_q^* \omega_{p'} V t'_{m-1} V^* \omega_{q'} = \delta_{p',q} \omega_p V t_{m-1} t'_{m-1} V^* \omega'_q$ and thus T_m is a C^*-algebra. The equation $V^* \omega_q^* \omega_p V = \delta_{p,q} I$ implies that $C^*(\{\omega_p V V^* \omega_q^* | 0 \le p, q \le l - 1\}) \cong M_l(\mathbb{C})$. It is clear that $V T_{m-1} V^*$ is a C^*-algebra, isomorphic to T_{m-1}. Therefore

$$C^*(\{\omega_p V t_{m-1} V^* \omega_q^* | 0 \le p, q \le l - 1, t_{m-1} \in T_{m-1}\}) \cong$$

$$C^*(\{\sum_{i=0}^{l-1} (\omega_i V t_{m-1} V^* \omega_i^*) | t_{m-1} \in T_{m-1}\}) \otimes C^*(\{\omega_p V V^* \omega_q^* | 0 \le p, q \le l - 1\})$$

$$\cong T_{m-1} \otimes M_l(\mathbb{C}) = M_l(T_{m-1}),$$

since $\sum_{i=0}^{l-1} (\omega_i V t_{m-1} V^* \omega_i^*)$ commutes with $\omega_p V V^* \omega_q^*$ for each $0 \le p, q \le l - 1$ and each $t_{m-1} \in T_{m-1}$. Taking limit $l \to \infty$ concludes the proof of the lemma. $\qquad \square$

From the proof of this lemma easily follows that T_m is the closed linear span of

$$\{\omega_{p_m} V \cdots V \omega_{p_1} V t_0 V^* \omega_{q_1}^* V^* \cdots V^* \omega_{q_m}^* | \ \omega_{p_1}, \ldots, \omega_{p_m}, \omega_{q_1}, \ldots, \omega_{q_m} \in \Omega, \ t_0 \in T_0\}.$$

This implies that $T_m \cdot T_l \subset T_m$ and $T_l \cdot T_m \subset T_m$ for each $m \geq l \geq 0$.

Now we introduce the following C^*-subalgebras of $C_Q^*(\Gamma)$: Define $B_0 = T_0$ and $B_m = C^*(B_{m-1} \cup T_m) = C^*(T_0 \cup \cdots \cup T_m)$. From what we said above is clear that T_m is an ideal of B_m. Therefore we have an extension

$$\text{(1)} \qquad 0 \longrightarrow T_m \xrightarrow{i_m} B_m \xrightarrow{p_m} B_m/T_m \longrightarrow 0,$$

where $i_m : T_m \to B_m$ is the inclusion map and $p_m : B_m \to B_m/T_m$ is the quotient map.

From [14, Theorem 3.1.7] (or [2, Corollary II.5.1.3]) follows that $B_m = B_{m-1} + T_m$ as a linear space. From [14, Remark 3.1.3] follows that the map $\pi_m : B_{m-1}/(B_{m-1} \cap T_m) \to B_m/T_m$ given by $b_{m-1} + B_{m-1} \cap T_m \mapsto b_{m-1} + T_m$ is an isomorphism ($b_{m-1} \in B_{m-1}$).

Define $\mathcal{I}_m \overset{def}{=} \langle V^m[\prod_{i=1}^{n}(I - V_iV_i^*)](V^*)^m \rangle_{T_m}$. Since $T_0 \cong C^*(\Gamma')$ from Theorem 1.2 follows that $\mathcal{I}_0$ is the unique nontrivial ideal of T_0 and it is isomorphic to $\mathcal{K}$. Then from Lemma 2.2 follows that $\mathcal{I}_m$ is the unique nontrivial ideal of T_m and it is isomorphic to $\mathcal{K}^{\otimes m} \otimes \mathcal{K}$. The ideal $\mathcal{I}_m$ can be described as the closed linear span of

$$\{\omega_{p_m}V \cdots V\omega_{p_1}V\iota_0 V^*\omega_{q_1}^* V^* \cdots V^*\omega_{q_m}^* \mid \omega_{p_1}, \ldots, \omega_{p_m}, \omega_{q_1}, \ldots, \omega_{q_m} \in \Omega, \iota_0 \in \mathcal{I}_0\}.$$

Therefore it is easy to see that $V^m(V^*)^m\mathcal{I}_mV^m(V^*)^m = V^m\mathcal{I}_0(V^*)^m$.

By the definition of $C_Q^*(\Gamma)$ we have $(I - VV^*)\prod_{i=1}^{n}(I - V_iV_i^*) = 0$ or $\prod_{i=1}^{n}(I - V_iV_i^*) = VV^*\prod_{i=1}^{n}(I - V_iV_i^*)$. Therefore using relations (2) and (3) from Theorem 1.1 we get

$$\prod_{i=1}^{n}(I - V_iV_i^*) = VV^*\prod_{i=1}^{n}(I - V_iV_i^*) = V\prod_{i=1}^{k}(I - V_iV_i^*)V^* \prod_{i=k+1}^{n}(I - V_iV_i^*) =$$
$$= V\prod_{i=1}^{k}(I - V_iV_i^*)V^* \in T_1.$$

It follows also that $V^mV\prod_{i=1}^{k}(I - V_iV_i^*)V^*(V^*)^m \in V^mT_1(V^*)^m \subset T_{m+1}$. It is easy to see that $T_{m+1} \cdot B_m \subset T_{m+1}$ and $B_m \cdot T_{m+1} \subset T_{m+1}$. This implies that $T_m \cap T_{m+1}$ is an ideal of T_m and that $B_m \cap T_{m+1}$ is an ideal of B_m. From this we can conclude that $\mathcal{I}_m \subset (T_m \cap T_{m+1})$ for each $m \in \mathbb{N}$. The reverse inclusion is also true:

Lemma 2.3. $B_m \cap T_{m+1} = \mathcal{I}_m$ for each $m \in \mathbb{N}_0$.

Proof. Since $\mathcal{I}_0$ is the unique nontrivial ideal of T_0 and since $T_0 \cap T_1$ is an ideal of T_0, then if we assume that $\mathcal{I}_0 \subsetneq T_0 \cap T_1$ it will follow that $T_0 = T_0 \cap T_1$. Then $I = 1_{T_0} = 1_{C_Q^*(\Gamma)} \in T_0 \subset T_1$. This will imply that $T_1 \cong \mathcal{K} \otimes T_0$ is a unital C^*-algebra which is a contradiction. Therefore $\mathcal{I}_0 = T_0 \cap T_1$.

It is easy to see that for each $m \in \mathbb{N}$ we have $V^m(V^*)^mT_mV^m(V^*)^m = V^mT_0(V^*)^m \cong T_0$ and that $V^m(V^*)^mT_{m+1}V^m(V^*)^m = V^mT_1(V^*)^m \cong T_1$. Thus if we assume that $T_m = T_m \cap T_{m+1}$ it will follow that $V^mT_0(V^*)^m \subset V^mT_1(V^*)^m$ and therefore that

$T_0 \subset T_1$. This is a contradiction with what we proved in the last paragraph. Therefore $T_m \cap T_{m+1} \subsetneq T_m$ and thus $T_m \cap T_{m+1} = \mathcal{I}_m$.

To conclude the proof of the lemma we have to show that $T_{m+1} \cap T_j = 0$ for each $0 \le j < m$. In this case we have once again that $T_{m+1} \cap T_j$ is an ideal of T_j. Therefore the assumption $T_{m+1} \cap T_j \ne 0$ implies that T_{m+1} contains the minimal nonzero ideal of T_j, $\mathcal{I}_j$. In particular $V^j \prod_{i=1}^{n} (I - V_i V_i^*)(V^*)^j = V^{j+1} \prod_{i=1}^{k} (I - V_i V_i^*)(V^*)^{j+1} \in T_{m+1}$. This implies

$$V^{j+1} \prod_{i=1}^{k} (I - V_i V_i^*)(V^*)^{j+1} = V^{j+1}(V^*)^{j+1} V^{j+1} \prod_{i=1}^{k} (I - V_i V_i^*)(V^*)^{j+1} V^{j+1}(V^*)^{j+1}$$

$$\in V^{j+1}(V^*)^{j+1} T_{m+1} V^{j+1}(V^*)^{j+1} = V^{j+1} T_{m-j}(V^*)^{j+1}.$$

Therefore $\prod_{i=1}^{k}(I - V_i V_i^*) \in T_{m-j}$. Since also $\prod_{i=1}^{k}(I - V_i V_i^*) \in T_0$, then the ideal $T_0 \cap T_{m-j}$ of T_0 contains $\prod_{i=1}^{k}(I - V_i V_i^*)$. We will show that $\prod_{i=1}^{k}(I - V_i V_i^*) \notin \mathcal{I}_0$ this will imply that $T_0 \subset T_{m-j}$ for $m - j > 0$ and therefore obtaining a contradiction with the fact that T_{m-j} is not unital for $m - j > 0$.

Suppose that $\prod_{i=1}^{k}(I - V_i V_i^*) \in \mathcal{I}_0$. Then since $T_0 = C^*(\Gamma')$ we have $Q_{\Gamma'}(\prod_{i=1}^{k}(I - V_i V_i^*)) = 0$. From the connectedness of $(\Gamma')^{\mathrm{opp}}$ follows that we can find j, $1 \le j \le k$ and l, $k < l \le n$ with $(j,l) \notin \Gamma'$. Then

$$0 = Q_{\Gamma'}(V_l^*) Q_{\Gamma'}(\prod_{i=1}^{k}(I - V_i V_i^*)) Q_{\Gamma'}(V_l) = Q_{\Gamma'}(V_l^*) Q_{\Gamma'}(\prod_{\substack{(i,l) \in \Gamma' \\ 1 \le i \le k}} (I - V_i V_i^*)) Q_{\Gamma'}(V_l) =$$

$$= Q_{\Gamma'}(V_l^* V_l) Q_{\Gamma'}(\prod_{\substack{(i,l) \in \Gamma' \\ 1 \le i \le k}} (I - V_i V_i^*)) = Q_{\Gamma'}(\prod_{\substack{(i,l) \in \Gamma' \\ 1 \le i \le k}} (I - V_i V_i^*)).$$

By repeating this argument finitely many times we will arrive at the equality $Q_{\Gamma'}(I) = 0$ which is a contradiction. Therefore $\prod_{i=1}^{k}(I - V_i V_i^*) \notin \mathcal{I}_0$. This completes the proof of the lemma. $\square$

This lemma shows that we have an extension

$$(2) \qquad\qquad 0 \to \mathcal{I}_{m-1} \xrightarrow{i'_m} B_{m-1} \xrightarrow{p'_m} B_{m-1}/\mathcal{I}_{m-1} \to 0,$$

where $i'_m : \mathcal{I}_{m-1} \to B_{m-1}$ is the inclusion map and $p'_m : B_{m-1} \to B_{m-1}/\mathcal{I}_{m-1}$ is the quotient map.

From equations (1) and (2) we have the commutative diagram with exact rows:

(3)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}_{m-1} & \xrightarrow{i'_m} & B_{m-1} & \xrightarrow{p'_m} & B_{m-1}/\mathcal{I}_{m-1} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle I'_m} & & \downarrow{\scriptstyle I_m} & & \cong \downarrow{\scriptstyle \pi_m} & & \\
0 & \longrightarrow & T_m & \xrightarrow{i_m} & B_m & \xrightarrow{p_m} & B_m/T_m & \longrightarrow & 0,
\end{array}
$$

where $I'_m : \mathcal{I}_{m-1} \to T_m$ and $I_m : B_{m-1} \to B_m$ are the inclusion maps.

Define $B \overset{def}{=} \overline{\bigcup_{i=0}^{\infty} B_i}^{\|\cdot\|} \subset C^*(\Gamma)$ or in other words $B \overset{def}{=} \varinjlim(B_m, I_m)$. Notice that if $t_m \in T_m$ then $V t_m V^* \in T_{m+1}$. Thus we have a well defined injective endomorphism $\beta : B \to B$ given by $b \mapsto V b V^*$.

Similarly to the Cuntz' construction from [7] we define $\tilde{B} \overset{def}{=} \varinjlim(B^m, \alpha_m)$ as the limit of the sequence (which is also a commutative diagram)
(4)
$$
\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\alpha_{-m-1}} & B^{-m} & \xrightarrow{\alpha_{-m}} & \cdots & \xrightarrow{\alpha_{-1}} & B^0 & \xrightarrow{\alpha_0} & B^1 & \xrightarrow{\alpha_1} & \cdots & \xrightarrow{\alpha_{m-1}} & B^m & \xrightarrow{\alpha_m} & \cdots \\
& & \downarrow{\scriptstyle j_{-m}}{\cong} & & & & \downarrow{\scriptstyle j_0}{\cong} & & \downarrow{\scriptstyle j_1}{\cong} & & & & \downarrow{\scriptstyle j_m}{\cong} & & \\
\cdots & \xrightarrow{\beta} & B & \xrightarrow{\beta} & \cdots & \xrightarrow{\beta} & B & \xrightarrow{\beta} & B & \xrightarrow{\beta} & \cdots & \xrightarrow{\beta} & B & \xrightarrow{\beta} & \cdots,
\end{array}
$$

where $j_m : B^m \to B$ are $*$-isomorphisms. Since $\tilde{B}$ is a limit C^*-algebra we have $*$-homomorphisms $\alpha^m : B^m \to \tilde{B}$, s.t. $\alpha^m = \alpha^{m+1} \circ \alpha_m$ for all $m \in \mathbb{Z}$.

Now we define a $*$-homorphism Φ of $\tilde{B}$ to itself, which is induced by "shift to the left" on (4). In other words if we have a stabilizing sequence $(b^m)_{m=-\infty}^{+\infty}$, where $b^m \in B^m$ for each m, then $\Phi((b^m)_{m=-\infty}^{+\infty}) = (j_m^{-1} \circ j_{m+1}(b^{m+1}))_{m=-\infty}^{+\infty}$. In particular for $b \in B$ the element $\alpha^m \circ j_m^{-1}(b)$ can be represented as the sequence $(0, \ldots, 0, 0, j_m^{-1}(b), \alpha_m \circ j_m^{-1}(b), \alpha_{m+1} \circ \alpha_m \circ j_m^{-1}(b), \ldots) = (0, \ldots, 0, 0, j_m^{-1}(b), j_{m+1}^{-1} \circ \beta(b), j_{m+2}^{-1} \circ \beta^2(b), \ldots)$ therefore $\Phi(\alpha^m \circ j_m^{-1}(b))$ can be represented as the sequence $(0, \ldots, 0, j_{m-1}^{-1}(b), j_m^{-1} \circ \beta(b), j_{m+1}^{-1} \circ \beta^2(b), \ldots)$. This shows that $\Phi(\alpha^m \circ j_m^{-1}(b)) = \alpha^m \circ j_m^{-1} \circ \beta(b)$. The extension of this map to the whole of $\tilde{B}$ (we call it Φ also) is a $*$-isomorphism, because Φ is isometric on the dense set of all stabilizing sequences (since j_m are all isomorphisms). Now let $\tilde{A}$ be the crossed product of $\tilde{B}$ by the automorphism Φ. We represent $\tilde{A}$ faithfully on a Hilbert space $\mathfrak{H}$ so that Φ is implemented by a unitary U on $\mathfrak{H}$: $\Phi(b) = U b U^*$ for $b \in \tilde{B}$. Then $\tilde{A} = C^*(\tilde{B} \cup \{U\})$. Every element of $\tilde{A}$ is a limit of elements of the form $\tilde{a} = \sum_{i=-N}^{N} b_i U^i = \sum_{i=-N}^{-1} U^i \bar{b}_i + b_0 + \sum_{i=1}^{N} b_i U^i$, with $b_i \in \tilde{B}$, where $\bar{b}_i = U^{-i} b_i U^i \in \tilde{B}$ for $i = -N, \ldots, -1$. Therefore the set of the elements of $\tilde{A}$ of the above form is dense in $\tilde{A}$.

Set $\tilde{P}_m \overset{def}{=} \alpha^m(1_{B^m}) \in \tilde{B}$ for each $m \in \mathbb{Z}$. Notice that $\alpha^m(1_{B^m}) = \alpha^m \circ j_m^{-1}(I) = \alpha^{m+1} \circ \alpha_m \circ j_m^{-1}(I) = \alpha^{m+1} \circ j_{m+1}^{-1}(\beta(I))$. By induction

$$
\tilde{P}_m = \alpha^{m+i} \circ j_{m+i}^{-1}(\beta^i(I)), \ m \in \mathbb{Z}, \ i \in \mathbb{N}.
$$

Therefore we can write

(5) $$\tilde{P}_m = \Phi^{-m}(\tilde{P}_0), \ m \in \mathbb{Z}.$$

Consider the C^*-algebra $\tilde{P}_0 \tilde{A} \tilde{P}_0$. Clearly $\tilde{P}_0 \tilde{B} \tilde{P}_0 \subset \tilde{P}_0 \tilde{A} \tilde{P}_0$. Since elements of the form $\tilde{a} = \sum\limits_{i=-N}^{-1} U^i b_i + b_0 + \sum\limits_{i=1}^{N} b_i U^i$ $(b_i \in \tilde{B})$ are dense in $\tilde{A}$, then elements of the form $\tilde{P}_0 \tilde{a} \tilde{P}_0 = \sum\limits_{i=-N}^{-1} \tilde{P}_0 U^i b_i \tilde{P}_0 + \tilde{P}_0 b_0 \tilde{P}_0 + \sum\limits_{i=1}^{N} \tilde{P}_0 b_i U^i \tilde{P}_0$ are dense in $\tilde{P}_0 \tilde{A} \tilde{P}_0$. It is easy to see that $U \tilde{P}_0 U^* = \Phi(\tilde{P}_0) < \tilde{P}_0$, so the range of $U \tilde{P}_0$ is contained in $\tilde{P}_0$ and therefore $\tilde{P}_0 U \tilde{P}_0 = U \tilde{P}_0$. Then

$$\tilde{P}_0 \tilde{a} \tilde{P}_0 = \sum_{i=-N}^{-1} \tilde{P}_0 U^i b_i \tilde{P}_0 + \tilde{P}_0 b_0 \tilde{P}_0 + \sum_{i=1}^{N} \tilde{P}_0 b_i U^i \tilde{P}_0 =$$

$$= \sum_{i=-N}^{-1} (\tilde{P}_0 U^i)(\tilde{P}_0 b_i \tilde{P}_0) + \tilde{P}_0 b_0 \tilde{P}_0 + \sum_{i=1}^{N} (\tilde{P}_0 b_i \tilde{P}_0)(U^i \tilde{P}_0).$$

This shows that if we set $S \overset{def}{=} U \tilde{P}_0$ then $\tilde{P}_0 \tilde{A} \tilde{P}_0 = C^*(\tilde{P}_0 \tilde{B} \tilde{P}_0 \cup \{S\})$. Let us also set $S_i \overset{def}{=} \alpha^0(j_0^{-1}(V_i)), \ i = 1, ..., n$.

It is easy to see that $\mathrm{Span}(\bigcup\limits_{l=0}^{\infty} T_l)$ is dense in B. Then it follows that $\mathrm{Span}(\bigcup\limits_{i=0}^{\infty} \alpha^i \circ j_i^{-1}(\bigcup\limits_{l=0}^{\infty} T_l))$ is dense in $\tilde{B}$. Therefore $\tilde{P}_0 \mathrm{Span}(\bigcup\limits_{i=0}^{\infty} \alpha^i \circ j_i^{-1}(\bigcup\limits_{l=0}^{\infty} T_l)) \tilde{P}_0 = \mathrm{Span}(\tilde{P}_0 \bigcup\limits_{i=0}^{\infty} \alpha^i \circ j_i^{-1}(\bigcup\limits_{l=0}^{\infty} T_l) \tilde{P}_0)$ is dense in $\tilde{P}_0 \tilde{B} \tilde{P}_0$. For each $i \in \mathbb{N}$ we have

$$\tilde{P}_0 \alpha^i \circ j_i^{-1}(\bigcup_{l=0}^{\infty} T_l) \tilde{P}_0 = \alpha^i \circ j_i^{-1}(\beta^i(I)) \alpha^i \circ j_i^{-1}(\bigcup_{l=0}^{\infty} T_l) \alpha^i \circ j_i^{-1}(\beta^i(I)) =$$

$$= \alpha^i \circ j_i^{-1}(\beta^i(I)(\bigcup_{l=0}^{\infty} T_l) \beta^i(I)) = \alpha^i \circ j_i^{-1}(V^i(V^*)^i(\bigcup_{l=0}^{\infty} T_l) V^i(V^*)^i) =$$

$$= \alpha^i \circ j_i^{-1}((V^i(V^*)^i)^2(\bigcup_{l=0}^{\infty} T_l)(V^i(V^*)^i)^2) \subset \alpha^i \circ j_i^{-1}((V^i(V^*)^i T_i)(\bigcup_{l=0}^{\infty} T_l)(T_i V^i(V^*)^i)) \subset$$

$$\subset \alpha^i \circ j_i^{-1}(V^i(V^*)^i(\bigcup_{l=i}^{\infty} T_l) V^i(V^*)^i) = \alpha^i \circ j_i^{-1}(V^i(\bigcup_{l=0}^{\infty} T_l)(V^*)^i) = \alpha^i \circ j_i^{-1}(\beta^i(\bigcup_{l=0}^{\infty} T_l))) =$$

$$= \alpha^i \circ \alpha_{i-1} \circ \alpha_{i-2} \circ \cdots \circ \alpha_1 \circ \alpha_0 \circ j_0^{-1}(\bigcup_{l=0}^{\infty} T_l) = \alpha^0 \circ j_0^{-1}(\bigcup_{l=0}^{\infty} T_l).$$

From this it follows that $\alpha_0 \circ j_0^{-1}(\mathrm{Span}(\bigcup\limits_{l=0}^{\infty} T_l))$ is dense in $\tilde{P}_0 \tilde{B} \tilde{P}_0$ and therefore also that $\alpha^0(B^0) = \tilde{P}_0 \tilde{B} \tilde{P}_0$. This shows that $\tilde{P}_0 \tilde{A} \tilde{P}_0 = C^*(\alpha_0 \circ j_0^{-1}(\bigcup\limits_{l=0}^{\infty} T_l) \cup \{S\})$.

Observe that

(6) $\quad S\alpha^0 \circ j_0^{-1}(b)S^* = U\tilde{P}_0\alpha^0 \circ j_0^{-1}(b)\tilde{P}_0U^* = U\alpha^0 \circ j_0^{-1}(b)U^* = \Phi(\alpha^0 \circ j_0^{-1}(b)) =$
$$= \alpha^0 \circ j_0^{-1} \circ \beta(b) = \alpha^0 \circ j_0^{-1}(VbV^*).$$

Since for every $m > 0$ T_m can be constructed from T_0 and "Ad(V)" equation (6) shows that $\tilde{P}_0\tilde{A}\tilde{P}_0 = C^*(\alpha_0 \circ j_0^{-1}(\bigcup_{l=0}^{\infty}T_l) \cup \{S\}) = C^*(\alpha_0 \circ j_0^{-1}(T_0) \cup \{S\}) = C^*(\{S_1, \ldots, S_n, S\})$.

We want to apply now Theorem 1.1 to the C^*-algebra $A \overset{def}{=} \tilde{P}_0\tilde{A}\tilde{P}_0$. $S_i = \alpha^0 \circ j_0^{-1}(V_i)$ are clearly isometries $(i = 1, \ldots, n)$. $S^*S = \tilde{P}_0U^*U\tilde{P}_0 = \tilde{P}_0$ and therefore S is also an isometry. Thus condition (1) holds.
It is clear from (6) that $SS^* = \alpha^0 \circ j_0^{-1}(VV^*)$. Therefore

$$0 = \alpha^0 \circ j_0^{-1}(0) = \alpha^0 \circ j_0^{-1}((I - VV^*)\prod_{i=1}^{n}(I - V_iV_i^*)) =$$

$$= (\tilde{P}_0 - \alpha^0 \circ j_0^{-1}(VV^*))\prod_{i=1}^{n}(\tilde{P}_0 - \alpha^0 \circ j_0^{-1}(V_iV_i^*)) = (\tilde{P}_0 - SS^*)\prod_{i=1}^{n}(\tilde{P}_0 - S_iS_i^*).$$

This proves that condition (4) holds.
Conditions (2) and (3) obviously hold for all pairs of isometries from $\{S_1, \ldots, S_n\}$. If $n \geq i > k$ then $S_iS_i^*SS^* = \alpha^0 \circ j_0^{-1}(V_iV_i^*VV^*) = 0$, so condition (3) holds also for all paris (S_i, S) with $k < i \leq n$. For $1 \leq i \leq k$ one has

$$SS_i = S\alpha^0 \circ j_0^{-1}(V_i) = S\alpha^0 \circ j_0^{-1}(V_i)S^*S = \Phi(\alpha^0 \circ j_0^{-1}(V_i))S = \alpha^0 \circ j_0^{-1}(VV_iV^*)S =$$
$$= \alpha^0 \circ j_0^{-1}(V_iVV^*)S = \alpha^0 \circ j_0^{-1}(V_i)\alpha^0 \circ j_0^{-1}(VV^*)S = S_iSS^*S = S_iS.$$

This shows that $SS_i = S_iS$. In the same way one can show that $SS_i^* = S_i^*S$. Therefore condition (4) holds for all pairs (S, S_i) with $1 \leq i \leq k$.
Applying Theorem 1.1 we get $A \cong C_Q^*(\Gamma)$. Obviously we also have $C_Q^*(\Gamma) \cong \tilde{P}_m\tilde{A}\tilde{P}_m$ for each $m \in \mathbb{Z}$.

We reming here (see [2, IV.3.1], [1, §22]) that each C^*-algebra in the small bootstrap class $\mathfrak{N}$ satisfies the Universal Coefficient Theorem. The small bootstrap class $\mathfrak{N}$ is the smallest class of C^*-algebras that satisfy:
(i) $\mathbb{C} \in \mathfrak{N}$.
(ii) $\mathfrak{N}$ is closed under stable isomorphism.
(iii) $\mathfrak{N}$ is closed under inductive limits.
(iv) $\mathfrak{N}$ is closed under crossed-products by $\mathbb{Z}$.
(v) If $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$ is an exact sequence, and two of $\mathfrak{I}, \mathfrak{A}, \mathfrak{A}/\mathfrak{I}$ are in $\mathfrak{N}$, so is the third.
The C^*-algebras in this class are all nuclear.
The following proposition holds:

Proposition 2.4. *In the above settings:* $\tilde{A} \cong \tilde{B} \rtimes_{\Phi} \mathbb{Z}$ *and* $A \cong C_Q^*(\Gamma)$ *is Morita equivalent to* $\tilde{A}$. *Both of the C^*-algebras A and $\tilde{A}$ are simple, belong to $\mathfrak{N}$ and*

$\mathbf{K}_*(\tilde{A}) = \mathbf{K}_*(A)$. *Also if we suppose that $[\tilde{P}_0]_0$ generates $\mathbf{K}_0(\tilde{A})$ then it follows that $[\tilde{P}_0]_0$ generates $\mathbf{K}_0(A)$.*

Proof. We showed above that $\tilde{P}_m\tilde{A}\tilde{P}_m \cong C_Q^*(\Gamma)$ for each $m \in \mathbb{Z}$. It is easy to see that $\tilde{A} = \overline{\bigcup_{m=0}^{\infty} \tilde{P}_m\tilde{A}\tilde{P}_m}$ and since each $\tilde{P}_m\tilde{A}\tilde{P}_m$ is simple from this follows that $\tilde{A}$ is simple too. Therefore every projection in $\tilde{A}$ is full. In particular $\tilde{P}_0$ is a full projection and therefore $A = \tilde{P}_0\tilde{A}\tilde{P}_0$ is a full corner of $\tilde{A}$ and is therefore Morita equivalent to $\tilde{A}$. It follows that A and $\tilde{A}$ are stably isomorphic (by Brown's Theorem [3]) and therefore $\mathbf{K}_*(A) = \mathbf{K}_*(\tilde{A})$.

If $\tilde{A}$ belongs to $\mathfrak{N}$ then from the definition follows that A also does since it is stably isomorphic to $\tilde{A}$.

To conclude the proof of the lemma it remains to show that starting from any finite graph G with G^{opp} connected and going through the above construction the C^*-algebra (let us denote it by $\tilde{A}_G$ - the analogue of $\tilde{A}$ for G) belongs to $\mathfrak{N}$. We will do this by using induction on the number of the vertices of G. If G has only two vertices and no edges then $C_Q^*(G) \cong \mathcal{O}_2$ and $C^*(G) \cong \mathcal{E}_2$ so the statement for this graph is true. Suppose that the statement is true for any graph G with at most $n \geq 2$ vertices such that its opposite graph G^{opp} is connected. In particular $C_Q^*(\Gamma')$ (and therefore also $C^*(\Gamma')$) belong to $\mathfrak{N}$. Then $T_0 \cong C^*(\Gamma')$ as constructed above also does. Since the bootstrap category is closed under stabilization, extensions, inductive limits and crossed products by $\mathbb{Z}$ we conclude using induction that the C^*-algebra $\tilde{A}$ is also nuclear and belong to the small bootstrap class (we use diagram (3) together with Lemma 2.2 and the fact that π_m is an isomorphism for all $m \in \mathbb{N}$). Finally as we showed in the last paragraph this implies that A belongs to $\mathfrak{N}$. This concludes the inductive step because $A \cong C_Q^*(\Gamma)$ and Γ is an arbitrary graph with $n+1$ vertices such that Γ^{opp} is connected.

The final statement of the proposition in obvious.

The proposition is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

3. The Computation of the $\mathbf{K}$-Theory

For a finite graph G with G^{opp} connected Crisp and Laca conjectured in [6] that the order of $[1_{C_Q^*(G)}]_0$ in $\mathbf{K}_0(C_Q^*(G))$ is $|\chi(G)|$, where $\chi(G)$ is the Euler characteristics of G. $\chi(G)$ is defined as

$$\chi(G) = 1 - \sum_{j=1}^{\infty}(-1)^{j-1} \times \{\text{ number of complete subgraphs of } G \text{ on } j \text{ vertices }\}.$$

We will use the settings from the previous section. Denote $P_m \overset{def}{=} V^m(V^*)^m$, $m \in \mathbb{N}_0$. Denote also $Q \overset{def}{=} \prod_{i=1}^{k}(I - V_iV_i^*)$. Let $\Gamma_k = \{(i,j)|\ 1 \leq i,\ j \leq k,\ (i,j) \in \Gamma'\} \subset \Gamma'$.

Since the vertex $n+1$ of Γ is connected with each of the vertices $1, \ldots, k$ and none of the others we have

$$\chi(\Gamma) = 1 - \sum_{j=1}^{n} (-1)^{j-1} \times \{ \text{ number of complete subgraphs of } \Gamma' \text{ on } j \text{ vertices } \} -$$

$$- (1 - \sum_{j=1}^{k} (-1)^{j-1} \times \{ \text{ number of complete subgraphs of } \Gamma_k \text{ on } j \text{ vertices } \}).$$

Therefore

$$(7) \qquad\qquad\qquad\qquad \chi(\Gamma) = \chi(\Gamma') - \chi(\Gamma_k).$$

The following lemma is based on the "Euler characteristics idea" and is essentially due to Crisp and Laca:

Lemma 3.1. *If E is a C^*-subalgebra of B that contains T_m (for $m \in \mathbb{N}_0$) we have*

$$(8) \qquad\qquad\qquad \chi(\Gamma')[P_m]_0 = [P_{m+1}Q]_0 \ \ (in \ \mathbf{K}_0(E)).$$

If E is a C^*-subalgebra of B that contains T_m and T_{m+1} (for $m \in \mathbb{N}_0$) we have

$$(9) \qquad\qquad\qquad \chi(\Gamma')[P_m]_0 = \chi(\Gamma_k)[P_{m+1}]_0 \ \ (in \ \mathbf{K}_0(E)).$$

If E is a C^*-subalgebra of B that contains T_{m+1} (for $m \in \mathbb{N}_0$) we have

$$(10) \qquad\qquad\qquad [P_{m+1}Q]_0 = \chi(\Gamma_k)[P_{m+1}]_0 \ \ (in \ \mathbf{K}_0(E)).$$

Proof. In the last section we showed that

$$(11) \qquad\qquad\qquad \prod_{i=1}^{n}(I - V_i V_i^*) = V \prod_{i=1}^{k}(I - V_i V_i^*)V^*.$$

Since $V^m \prod_{i=1}^{n}(I - V_i V_i^*)(V^*)^m = \prod_{i=1}^{n}(V^m(V^*)^m - V^m V_i V_i^*(V^*)^m)$ then by multiplying equation (11) by V^m on the left and by $(V^*)^m$ on the right we get

$$\prod_{i=1}^{n}(V^m(V^*)^m - V^m V_i V_i^*(V^*)^m) = V^{m+1}\prod_{i=1}^{k}(I - V_i V_i^*)(V^*)^{m+1} =$$

$$= V^{m+1}(V^*)^{m+1}Q.$$

This equation is actually three equations which hold in certain C^*-subalgebras of B. We record them here:

If E is a C^*-subalgebra of B that contains T_m (for $m \in \mathbb{N}_0$) we have

$$(12) \qquad\qquad \prod_{i=1}^{n}(V^m(V^*)^m - V^m V_i V_i^*(V^*)^m) = V^{m+1}(V^*)^{m+1}Q.$$

If E is a C^*-subalgebra of B that contains T_m and T_{m+1} (for $m \in \mathbb{N}_0$) we have

$$(13) \qquad \prod_{i=1}^{n}(V^m(V^*)^m - V^m V_i V_i^*(V^*)^m) = V^{m+1}\prod_{i=1}^{k}(I - V_i V_i^*)(V^*)^{m+1}.$$

If E is a C^*-subalgebra of B that contains T_{m+1} (for $m \in \mathbb{N}_0$) we have

$$(14) \qquad V^{m+1}\prod_{i=1}^{k}(I - V_i V_i^*)(V^*)^{m+1} = V^{m+1}(V^*)^{m+1}Q.$$

Note that if E is an appropriate C^*-subalgebra of B then for each projection P that commutes with $V_1 V_1^*$ we have $[V^m P(V^*)^m - V^m P V_1 V_1^* (V^*)^m]_0 = [V^m P(V^*)^m]_0 - [V^m P V_1 V_1^* (V^*)^m]_0$. Suppose by induction that for some $n > l \geq 1$ if P is a projection that commutes with $V_1 V_1^*, \ldots, V_l V_l^*$ we have

$$(15) \quad [V^m \prod_{i=1}^{l}(P - PV_i V_i^*)(V^*)^m]_0 =$$

$$[V^m P(V^*)^m]_0 - \sum_{i=1}^{l}[V^m P V_i V_i^* (V^*)^m]_0 +$$

$$+ \sum_{j=2}^{l}(-1)^j \Big(\sum_{\substack{1 \leq i_1 < \cdots < i_j \leq l \\ (i_s, i_t) \in \Gamma', 1 \leq s < t \leq j}} [V^m P V_{i_1} \cdots V_{i_j} V_{i_j}^* \cdots V_{i_1}^* (V^*)^m]_0 \Big).$$

We know that $V_{l+1} V_{l+1}^*$ commutes with each of $V_1 V_1^*, \ldots, V_l V_l^*$. If P commutes with $V_1 V_1^*, \ldots, V_{l+1} V_{l+1}^*$ then we can apply (15) to the family $V_1 V_1^*, \ldots, V_l V_l^*$ and the projection $PV_{l+1} V_{l+1}^*$ to obtain the following equation:

$$[V^m V_{l+1} V_{l+1}^* \prod_{i=1}^{l}(P - PV_i V_i^*)(V^*)^m]_0 = [V^m \prod_{i=1}^{l}(PV_{l+1} V_{l+1}^* - PV_{l+1} V_{l+1}^* V_i V_i^*)(V^*)^m]_0 =$$

$$= [V^m P V_{l+1} V_{l+1}^* (V^*)^m]_0 - \sum_{i=1}^{l}[V^m P V_{l+1} V_{l+1}^* V_i V_i^* (V^*)^m]_0 +$$

$$+ \sum_{j=2}^{l}(-1)^j \Big(\sum_{\substack{1 \leq i_1 < \cdots < i_j \leq l \\ (i_s, i_t) \in \Gamma', 1 \leq s < t \leq j}} [V^m P V_{l+1} V_{l+1}^* V_{i_1} \cdots V_{i_j} V_{i_j}^* \cdots V_{i_1}^* (V^*)^m]_0 \Big).$$

Now since $V^m V_{l+1} V_{l+1}^* \prod_{i=1}^{l}(P - PV_i V_i^*)(V^*)^m < V^m \prod_{i=1}^{l}(P - PV_i V_i^*)(V^*)^m$ it is easy to see that we have

$$[V^m(P - PV_{l+1}V_{l+1}^*)\prod_{i=1}^{l}(P - PV_i V_i^*)(V^*)^m]_0 =$$

$$= [V^m\prod_{i=1}^{l}(P - PV_i V_i^*)(V^*)^m - V^m V_{l+1} V_{l+1}^*\prod_{i=1}^{l}(P - PV_i V_i^*)(V^*)^m]_0 =$$

$$= [V^m\prod_{i=1}^{l}(P - PV_i V_i^*)(V^*)^m]_0 - [V^m V_{l+1} V_{l+1}^*\prod_{i=1}^{l}(P - PV_i V_i^*)(V^*)^m]_0 =$$

$$= [V^m P(V^*)^m]_0 - \sum_{i=1}^{l}[V^m PV_i V_i^*(V^*)^m]_0 +$$

$$+ \sum_{j=2}^{l}(-1)^j \sum_{\substack{1\leq i_1 < \cdots < i_j \leq l \\ (i_s, i_t)\in\Gamma', 1\leq s<t\leq j}} [V^m PV_{i_1}\cdots V_{i_j} V_{i_j}^*\cdots V_{i_1}^*(V^*)^m]_0 - [V^m PV_{l+1}V_{l+1}^*(V^*)^m]_0 +$$

$$+ \sum_{i=1}^{l}[V^m PV_{l+1}V_{l+1}^* V_i V_i^*(V^*)^m]_0 -$$

$$- \sum_{j=2}^{l}(-1)^j \sum_{\substack{1\leq i_1 < \cdots < i_j \leq l \\ (i_s, i_t)\in\Gamma', 1\leq s<t\leq j}} [V^m PV_{l+1}V_{l+1}^* V_{i_1}\cdots V_{i_j} V_{i_j}^*\cdots V_{i_1}^*(V^*)^m]_0 =$$

$$= [V^m P(V^*)^m]_0 - \sum_{i=1}^{l+1}[V^m PV_i V_i^*(V^*)^m]_0 +$$

$$+ \sum_{j=2}^{l+1}(-1)^j \sum_{\substack{1\leq i_1 < \cdots < i_j \leq l+1 \\ (i_s, i_t)\in\Gamma', 1\leq s<t\leq j}} [V^m PV_{i_1}\cdots V_{i_j} V_{i_j}^*\cdots V_{i_1}^*(V^*)^m]_0.$$

Then by induction follows that for $l = k$ or $l = n$ we get

$$[\prod_{i=1}^{l}(V^m(V^*)^m - V^m V_i V_i^*(V^*)^m)]_0 =$$

$$= [I]_0 - \sum_{i=1}^{l}[V^m V_i V_i^*(V^*)^m]_0 + \sum_{j=2}^{l}(-1)^j \sum_{\substack{1\leq i_1 < \cdots < i_j \leq l \\ (i_s, i_t)\in\Gamma', 1\leq s<t\leq j}} [V^m V_{i_1}\cdots V_{i_j} V_{i_j}^*\cdots V_{i_1}^*(V^*)^m]_0.$$

Combining the last equation with equations (12), (13) and (14) we obtain the following equations:

If E is a C^*-subalgebra of B that contains T_m (for $m \in \mathbb{N}_0$) we have

$$(16) \quad [V^m(V^*)^m]_0 - \sum_{i=1}^n [V^m V_i V_i^*(V^*)^m]_0 +$$

$$+ \sum_{j=2}^n (-1)^j (\sum_{\substack{1 \le i_1 < \cdots < i_j \le n \\ (i_s, i_t) \in \Gamma', 1 \le s < t \le j}} [V^m V_{i_1} \cdots V_{i_j} V_{i_j}^* \cdots V_{i_1}^* (V^*)^m]_0) =$$

$$= [V^{m+1}(V^*)^{m+1} Q]_0.$$

If E is a C^*-subalgebra of B that contains T_m and T_{m+1} (for $m \in \mathbb{N}_0$) we have

$$(17) \quad [V^m(V^*)^m]_0 - \sum_{i=1}^n [V^m V_i V_i^*(V^*)^m]_0 +$$

$$+ \sum_{j=2}^n (-1)^j (\sum_{\substack{1 \le i_1 < \cdots < i_j \le n \\ (i_s, i_t) \in \Gamma', 1 \le s < t \le j}} [V^m V_{i_1} \cdots V_{i_j} V_{i_j}^* \cdots V_{i_1}^* (V^*)^m]_0) =$$

$$= [V^{m+1}(V^*)^{m+1}]_0 - \sum_{i=1}^k [V^{m+1} V_i V_i^*(V^*)^{m+1}]_0 +$$

$$+ \sum_{j=2}^k (-1)^j (\sum_{\substack{1 \le i_1 < \cdots < i_j \le k \\ (i_s, i_t) \in \Gamma_k, 1 \le s < t \le j}} [V^{m+1} V_{i_1} \cdots V_{i_j} V_{i_j}^* \cdots V_{i_1}^* (V^*)^{m+1}]_0).$$

If E is a C^*-subalgebra of B that contains T_{m+1} (for $m \in \mathbb{N}_0$) we have

$$(18) \quad [V^{m+1}(V^*)^{m+1}]_0 - \sum_{i=1}^k [V^{m+1} V_i V_i^*(V^*)^{m+1}]_0 +$$

$$+ \sum_{j=2}^k (-1)^j (\sum_{\substack{1 \le i_1 < \cdots < i_j \le k \\ (i_s, i_t) \in \Gamma_k, 1 \le s < t \le j}} [V^{m+1} V_{i_1} \cdots V_{i_j} V_{i_j}^* \cdots V_{i_1}^* (V^*)^{m+1}]_0) =$$

$$= [V^{m+1}(V^*)^{m+1} Q]_0.$$

It is easy to see that in each C^*-subalgebra of B that contains T_m the projection $V^m V_{i_1} \cdots V_{i_j} V_{i_j}^* \cdots V_{i_1}^* (V^*)^m$ is Murray - von Neumann equivalent to $V^m(V^*)^m$ via the partial isometry $V^m V_{i_1} \cdots V_{i_j} (V^*)^m \in T_m$, where $\{i_1, \ldots, i_j\} \subset \{1, \ldots, n\}$. This observation together with equations (16), (17) and (18) give:

If E is a C^*-subalgebra of B that contains T_m we have

$$(19) \qquad [P_m]_0 - \sum_{i=1}^n [P_m]_0 + \sum_{j=2}^n (-1)^j (\sum_{\substack{1 \le i_1 < \cdots < i_j \le n \\ (i_s, i_t) \in \Gamma', 1 \le s < t \le j}} [P_m]_0) = [P_{m+1} Q]_0.$$

If E is a C^*-subalgebra of B that contains T_m and T_{m+1} then we have

$$(20) \quad [P_m]_0 - \sum_{i=1}^{n}[P_m]_0 + \sum_{j=2}^{n}(-1)^j(\sum_{\substack{1\leq i_1<\cdots<i_j\leq n \\ (i_s,i_t)\in\Gamma',1\leq s<t\leq j}}[P_m]_0) =$$

$$= [P_{m+1}]_0 - \sum_{i=1}^{k}[P_{m+1}]_0 + \sum_{j=2}^{k}(-1)^j(\sum_{\substack{1\leq i_1<\cdots<i_j\leq k \\ (i_s,i_t)\in\Gamma_k,1\leq s<t\leq j}}[P_{m+1}]_0).$$

If E is a C^*-subalgebra of B that contains T_{m+1} we have

$$(21) \quad [P_{m+1}]_0 - \sum_{i=1}^{k}[P_{m+1}]_0 + \sum_{j=2}^{k}(-1)^j(\sum_{\substack{1\leq i_1<\cdots<i_j\leq k \\ (i_s,i_t)\in\Gamma_k,1\leq s<t\leq j}}[P_{m+1}]_0) = [P_{m+1}Q]_0.$$

The last three equations are what we had to prove. $\square$

Remark 3.2. *It also follows from this lemma that if we denote the isometries that generate $C^*(\Gamma)$ by $\tilde{V},\tilde{V}_1,\ldots,\tilde{V}_n$, then*

$$[(I-\tilde{V}\tilde{V}^*)\prod_{i=1}^{n}(I-\tilde{V}_i\tilde{V}_i^*)]_0 = \chi(\Gamma)[I]_0 \ (in \ K_0(C^*(\Gamma))).$$

Therefore in the extenstion

$$(22) \qquad 0 \rightarrow \langle(I-\tilde{V}\tilde{V}^*)\prod_{i=1}^{n}(I-\tilde{V}_i\tilde{V}_i^*)\rangle \xrightarrow{I_\Gamma} C^*(\Gamma) \xrightarrow{Q_\Gamma} C_Q^*(\Gamma) \rightarrow 0$$

the map $I_{\Gamma*}$ on $\mathbf{K}_0$ is given by

$$[(I-\tilde{V}\tilde{V}^*)\prod_{i=1}^{n}(I-\tilde{V}_i\tilde{V}_i^*)]_0 \mapsto \chi(\Gamma)[I]_0.$$

Now we can state and prove the following

Proposition 3.3. *Suppose that G is a finite graph with at least two vertices and suppose that G^{opp} is connected. Then*

$$(23) \quad \mathbf{K}_0(C_Q^*(G)) = \begin{cases} \mathbb{Z}_{|\chi(G)|}, & if \ \chi(G) \neq 0, \\ \mathbb{Z}, & if \ \chi(G) = 0, \end{cases} \quad \mathbf{K}_1(C_Q^*(G)) = \begin{cases} 0, & if \ \chi(G) \neq 0, \\ \mathbb{Z}, & if \ \chi(G) = 0, \end{cases}$$

and $[1_{C_Q^*(G)}]_0$ generates $\mathbf{K}_0(C_Q^*(G))$ in all cases.

Moreover $\mathbf{K}_0(C^*(G)) = \mathbb{Z}$, $\mathbf{K}_1(C^*(G)) = 0$ and $[1_{C^*(G)}]_0$ generates $\mathbf{K}_0(C^*(G))$ in all cases.

Proof. We will use induction on the number of vertices of G. If G has two vertices (and no edges) then $C_Q^*(G) = \mathcal{O}_2$ and $C^*(G) = \mathcal{E}_2$ and in this case certainly the statement is true. Suppose that the statement is true for all graphs G with at most $n \geq 2$ vertices and with G^{opp} connected. The graph Γ considered above was a randomly

chosen graph with $n+1$ vertices and with the property that Γ^{opp} is connected. If we show that the statement holds for Γ than this will prove the statement by induction.

We note that from Lemma 2.2 and the assumption follows that $\mathbf{K}_0(T_m) = \mathbb{Z}[P_m]_0$ and $\mathbf{K}_1(T_m) = 0$ for all $m \in \mathbb{N}_0$. Also since $\mathcal{I}_m \cong \mathcal{K}$ we have $\mathbf{K}_0(\mathcal{I}_m) = \mathbb{Z}[P_m Q]_0$ and $\mathbf{K}_1(\mathcal{I}_m) = 0$ for all $m \in \mathbb{N}_0$. Finally we remind that π_m is an isomorphism for all $m \in \mathbb{N}_0$.

From the **K**-theory six term eact sequences for the two exact rows of (3) we have the following commutative diagram:

(24)

$$\mathbf{K}_0(\mathcal{I}_{m-1}) \quad \overset{i'_{m*}}{\to} \quad \mathbf{K}_0(B_{m-1}) \quad \overset{p'_{m*}}{\to} \quad \mathbf{K}_0(\tfrac{B_{m-1}}{\mathcal{I}_{m-1}})$$

$$\searrow I'_{m*} \qquad\qquad \downarrow I_{m*} \qquad\qquad \overset{\cong}{\nearrow} \pi_{m*}$$

$$\mathbf{K}_0(T_m) \quad \overset{i_{m*}}{\to} \quad \mathbf{K}_0(B_m) \quad \overset{p_{m*}}{\to} \quad \mathbf{K}_0(\tfrac{B_m}{T_m})$$

$$\uparrow \gamma_m^{\mathrm{ind}} \qquad\qquad \uparrow \delta_m^{\mathrm{ind}} \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow$$

$$\mathbf{K}_1(\tfrac{B_m}{T_m}) \quad \overset{p_{m*}}{\leftarrow} \quad \mathbf{K}_1(B_m) \quad \leftarrow \quad 0$$

$$\overset{\cong}{\nearrow} \pi_{m*} \qquad\qquad \uparrow I_{m*} \qquad\qquad\qquad \searrow$$

$$\mathbf{K}_1(\tfrac{B_{m-1}}{\mathcal{I}_{m-1}}) \quad \overset{p'_{m*}}{\leftarrow} \quad \mathbf{K}_1(B_{m-1}) \quad \leftarrow \quad 0,$$

where γ_m^{ind} and δ_m^{ind} are the index maps for the corresponding six term exact sequences.

Since $\mathcal{I}_{m-1}$ is generated by $P_m Q$ from Lemma 3.1 follows that the map $i_{m*} : \mathbf{K}_0(\mathcal{I}_{m-1}) \to \mathbf{K}_0(B_{m-1})$ is induced by $[P_m Q]_{\mathbf{K}_0(\mathcal{I}_{m-1})} \mapsto \chi(\Gamma')[P_{m-1}]_{\mathbf{K}_0(B_{m-1})}$. Also the map $I'_{m*} : \mathbf{K}_0(\mathcal{I}_{m-1}) \to \mathbf{K}_0(T_m)$ is induced by $[P_m Q]_{\mathbf{K}_0(\mathcal{I}_{m-1})} \mapsto \chi(\Gamma_k)[P_m]_{\mathbf{K}_0(T_m)}$.

When we "apply" β to equations (1) and (2) we obtain the following commutative diagrams with exact rows:

(25)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{I}_{m-1} & \overset{i'_m}{\longrightarrow} & B_{m-1} & \overset{p'_m}{\longrightarrow} & B_{m-1}/\mathcal{I}_{m-1} & \longrightarrow & 0 \\
 & & \beta \downarrow & & \beta \downarrow & & \bar\beta \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_m & \overset{i'_{m+1}}{\longrightarrow} & B_m & \overset{p'_{m+1}}{\longrightarrow} & B_m/\mathcal{I}_m & \longrightarrow & 0
\end{array}$$

and

(26)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_m & \overset{i_m}{\longrightarrow} & B_m & \overset{p_m}{\longrightarrow} & B_m/T_m & \longrightarrow & 0 \\
 & & \beta \downarrow & & \beta \downarrow & & \bar\beta \downarrow & & \\
0 & \longrightarrow & T_{m+1} & \overset{i_{m+1}}{\longrightarrow} & B_{m+1} & \overset{p_{m+1}}{\longrightarrow} & B_{m+1}/T_{m+1} & \longrightarrow & 0,
\end{array}$$

where $\bar{\beta}$ and $\bar{\bar{\beta}}$ are induced by β on the above quotients.

We can now start examining the five different cases depending on $\chi(\Gamma')$ and $\chi(\Gamma_k)$:
(case **I**): $\chi(\Gamma') = 0$ and $\chi(\Gamma_k) = 0$.
By assumption $i'_{m*} = 0 = I'_{m*}$. From (24) is easy to see that $\delta^{\text{ind}}_m = 0$. Therefore (24) splits into two:

(27)
$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{i'_{m*}=0} & \mathbf{K}_0(B_{m-1}) & \xrightarrow[\cong]{p'_{m*}} & \mathbf{K}_0(B_{m-1}/\mathcal{I}_{m-1}) & \longrightarrow & 0 \\
& & \Big\downarrow I_{m*} & & \Big\downarrow \pi_{m*} \; \cong & & \\
\xrightarrow{\delta^{\text{ind}}_m=0} & \mathbf{K}_0(T_m) & \xrightarrow{i_{m*}} & \mathbf{K}_0(B_m) & \xrightarrow{p_{m*}} & \mathbf{K}_0(B_m/T_m) & \longrightarrow \quad 0,
\end{array}
$$

(28)
$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{K}_1(B_{m-1}) & \xrightarrow{p'_{m*}} & \mathbf{K}_1(B_{m-1}/\mathcal{I}_{m-1}) & \xrightarrow{\gamma^{\text{ind}}_m} & \mathbf{K}_0(\mathcal{I}_{m-1}) & \xrightarrow{i'_{m*}=0} & \cdots \\
& & \Big\downarrow I_{m*} & & \Big\downarrow \pi_{m*} \; \cong & & & & \\
0 & \longrightarrow & \mathbf{K}_1(B_m) & \xrightarrow[\cong]{p_{m*}} & \mathbf{K}_1(B_m/T_m) & \xrightarrow{\delta^{\text{ind}}_m=0} & \cdots & &
\end{array}
$$
.

Suppose by induction that $\mathbf{K}_0(B_{m-1}) = \mathbb{Z}[P_0]_0 \oplus \cdots \oplus \mathbb{Z}[P_{m-1}]_0$. Notice that for $m = 1$ we have $\mathbf{K}_0(B_0) = \mathbb{Z}[P_0]_0$. Then from (27) follows that $\mathbf{K}_0(B_m) = I_{m*}(\mathbf{K}_0(B_{m-1})) \oplus i_{m*}(\mathbf{K}_0(T_m))$ since all extensions of free abelian groups are trivial. Noting that $\mathbf{K}_0(T_m) = \mathbb{Z}[P_m]_0$ concludes the induction. Therefore $\mathbf{K}_0(B_m) = \mathbb{Z}[P_0]_0 \oplus \cdots \oplus \mathbb{Z}[P_m]_0$ for each $m \in \mathbb{N}$. Notice that we can write $\mathbf{K}_0(B_m) = \mathbb{Z}[P_0]_0 \oplus \cdots \oplus \mathbb{Z}\beta^m_*([P_0]_0)$

Suppose by induction that

$$\mathbf{K}_1(B_{m-1}) = \mathbb{Z}(p_{1*})^{-1} \circ \pi_{1*} \circ (\gamma^{\text{ind}}_1)^{-1}([P_1 Q]_0) \oplus \cdots$$
$$\cdots \oplus \mathbb{Z}(p_{m-1*})^{-1} \circ \pi_{m-1*} \circ (\gamma^{\text{ind}}_{m-1})^{-1}([P_{m-1} Q]_0).$$

This is trivially true for $m = 1$. From (28) we see that $\mathbf{K}_1(B_m) = (p_{m*})^{-1} \circ \pi_{m*}(\mathbf{K}_1(B_{m-1}/\mathcal{I}_{m-1}))$. Since all groups are free abelian all the extensions are trivial and therefore $\mathbf{K}_1(B_m) = I_{m*}(\mathbf{K}(B_{m-1})) \oplus \mathbb{Z}(p_{m*})^{-1} \circ \pi_{m*} \circ (\gamma^{\text{ind}}_m)^{-1}([P_m Q]_0)$. This concludes the induction. From the functoriality of the index map and from equations (25) and (26) follows that $(p_{m*})^{-1} \circ \pi_{m*} \circ (\gamma^{\text{ind}}_m)^{-1}([P_m Q]_0) = (p_{m*})^{-1} \circ \pi_{m*} \circ (\gamma^{\text{ind}}_m)^{-1} \circ \beta_*([P_{m-1} Q]_0) = \beta_* \circ (p_{m-1*})^{-1} \circ \pi_{m-1*} \circ (\gamma^{\text{ind}}_{m-1})^{-1}([P_{m-1} Q]_0)$. Therefore we can write

$$\mathbf{K}_1(B_m) = \mathbb{Z}(p_{1*})^{-1} \circ \pi_{1*} \circ (\gamma^{\text{ind}}_1)^{-1}([P_1 Q]_0) \oplus \cdots$$
$$\cdots \oplus \mathbb{Z}\beta^{m-1}_* \circ (p_{1*})^{-1} \circ \pi_{1*} \circ (\gamma^{\text{ind}}_1)^{-1}([P_1 Q]_0).$$

If $u \in B_1$ is a unitary with $[u]_1 = (p_{1*})^{-1} \circ \pi_{1*} \circ (\gamma^{\text{ind}}_1)^{-1}([P_1 Q]_0)$ then we can write

$$\mathbf{K}_1(B_m) = \mathbb{Z}[u]_1 \oplus \cdots \oplus \mathbb{Z}\beta^{m-1}_*([u]_1).$$

From this we get $\mathbf{K}_0(B) = \overset{\infty}{\underset{i=0}{\oplus}} \mathbb{Z}\beta_*^i([I]_0)$ and $\mathbf{K}_1(B) = \overset{\infty}{\underset{i=1}{\oplus}} \mathbb{Z}\beta_*^{i-1}([u]_1)$.

Let $\tilde{u} = \alpha^0 \circ j_0(u) \in \tilde{B}$. Then it is easy to see that $\mathbf{K}_0(\tilde{B}) = \overset{\infty}{\underset{i=-\infty}{\oplus}} \mathbb{Z}\Phi_*^i([\tilde{P}_0]_0))$ and $\mathbf{K}_1(\tilde{B}) = \overset{\infty}{\underset{i=-\infty}{\oplus}} \mathbb{Z}\Phi_*^{i-1}([\tilde{u}]_1)$.

The Pimsner-Voiculescu gives

$$\begin{array}{ccccc}
\overset{\infty}{\underset{i=-\infty}{\oplus}} \mathbb{Z}\Phi_*^i([\tilde{P}_0]_0)) & \xrightarrow{\mathrm{id}_* - \Phi_*} & \overset{\infty}{\underset{i=-\infty}{\oplus}} \mathbb{Z}\Phi_*^i([\tilde{P}_0]_0)) & \longrightarrow & \mathbf{K_0}(\tilde{A}) \\
\big\uparrow & & & & \big\downarrow \\
\mathbf{K_1}(\tilde{A}) & \longleftarrow & \overset{\infty}{\underset{i=-\infty}{\oplus}} \mathbb{Z}\Phi_*^{i-1}([\tilde{u}]_1) & \xleftarrow{\mathrm{id}_* - \Phi_*} & \overset{\infty}{\underset{i=-\infty}{\oplus}} \mathbb{Z}\Phi_*^{i-1}([\tilde{u}]_1).
\end{array}$$

From this we can conclude that $\mathbf{K}_0(\tilde{A}) = \mathbb{Z}[\tilde{P}_0]_0$, $\mathbf{K}_1(\tilde{A}) = \mathbb{Z}$. From Proposition 2.4 follows that $\mathbf{K}_0(\tilde{A}) \cong \mathbf{K}_0(C_Q^*(\Gamma)) = \mathbb{Z}$ and $\mathbf{K}_1(\tilde{A}) \cong \mathbf{K}_1(C_Q^*(\Gamma)) = \mathbb{Z}$ and that $[1_{C_Q^*(\Gamma)}]_0$ generates $\mathbf{K}_0(C_Q^*(\Gamma))$.

From Remark 3.2 follows that in the extension (22) the map $I_{\Gamma*}$ on $\mathbf{K}_0$ is zero. This shows that $\mathbf{K}_0(C^*(\Gamma)) = \mathbb{Z}[1_{C^*(\Gamma)}]_0$ and $\mathbf{K}_1(C^*(\Gamma)) = 0$.

This concludes the proof of (case **I**).

(case **II**): $\chi(\Gamma') \neq 0$ and $\chi(\Gamma_k) = 0$.

By assumption $\mathbf{K}_0(B_0) = \mathbb{Z}[P_0]_0$, $\mathbf{K}_0(B_0/\mathcal{I}_0) = \mathbb{Z}_{|\chi(\Gamma')|}p'_{1*}([P_0]_0)$, $\mathbf{K}_1(B_0) = 0$ and $\mathbf{K}_1(B_0/\mathcal{I}_0) = 0$.

Suppose by induction that

$$\mathbf{K}_0(B_{m-1}) = \mathbb{Z}[P_{m-1}]_0 \oplus \mathbb{Z}_{|\chi(\Gamma')|}[P_{m-2}]_0 \oplus \cdots \oplus \mathbb{Z}_{|\chi(\Gamma')|}[P_0]_0,$$

$$\mathbf{K}_0(B_{m-1}/\mathcal{I}_{m-1}) = \mathbb{Z}_{|\chi(\Gamma')|}p'_{m*}([P_{m-1}]_0) \oplus \mathbb{Z}_{|\chi(\Gamma')|}p'_{m*}([P_{m-2}]_0) \oplus \cdots \oplus \mathbb{Z}_{|\chi(\Gamma')|}p'_{m*}([P_0]_0),$$

$\mathbf{K}_1(B_{m-1}) = 0$ and $\mathbf{K}_1(B_{m-1}/\mathcal{I}_{m-1}) = 0$.

Then from diagram (24) immediately follows that $\mathbf{K}_1(B_m) = 0$ and $\mathbf{K}_1(B_m/T_m) = 0$.

Then (24) reduces to the following commutative diagram with exact rows:

$$(29) \quad \begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{K}_0(\mathcal{I}_{m-1}) & \xrightarrow{i'_{m*}} & \mathbf{K}_0(B_{m-1}) & \xrightarrow{p'_{m*}} & \mathbf{K}_0(B_{m-1}/\mathcal{I}_{m-1}) & \longrightarrow & 0 \\
& & I'_{m*}=0 \big\downarrow & & I_{m*} \big\downarrow & & \cong \big\downarrow \pi_{m*} & & \\
0 & \longrightarrow & \mathbf{K}_0(T_m) & \xrightarrow{i_{m*}} & \mathbf{K}_0(B_m) & \xrightarrow{p_{m*}} & \mathbf{K}_0(B_m/T_m) & \longrightarrow & 0.
\end{array}$$

From Lemma 3.1 we have that $\chi(\Gamma')[P_l]_0 = 0$ in $\mathbf{K}_0(B_m)$ for $l = 0, \ldots, m-1$. Since π_{m*} is an isomorphism then by the inducton hypothesis and (29) it is easy to see that p_{m*} restricted to $\mathcal{G} = \langle [P_0]_0, \ldots, [P_{m-1}]_0 \rangle_{\mathbf{K}_0(B_m)}$ is an isomorphism. This fact also implies that there are no relations between $[P_m]_0$ and $\mathcal{G}$ (since the bottom row of (29) is exact). Since i_{m*} is injective then $[P_m]_0$ in of infinite order in $\mathbf{K}_0(B_m)$. Clearly $\mathbf{K}_0(B_m)$ is generated by $[P_m]_0$ and $\mathcal{G}$. Therefore

$$\mathbf{K}_0(B_m) = \mathbb{Z}[P_m]_0 \oplus \mathbb{Z}_{|\chi(\Gamma')|}[P_{m-1}]_0 \oplus \cdots \oplus \mathbb{Z}_{|\chi(\Gamma')|}[P_0]_0.$$

From the following six term exact sequence

$$\mathbf{K_0}(\mathcal{I}_m) \xrightarrow{i'_{m*}} \mathbf{K_0}(B_m) \xrightarrow{p'_{m*}} \mathbf{K_0}(B_m/\mathcal{I}_m)$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathbf{K_1}(B_m/\mathcal{I}_m) \xleftarrow{p'_{m*}} \mathbf{K_1}(B_m) \xleftarrow{i'_{m*}} \mathbf{K_1}(\mathcal{I}_m)$$

and the fact that i'_{m*} is given by $[P_{m+1}Q]_0 \mapsto \chi(\Gamma')[P_m]_0$ we easily get that
$\mathbf{K_1}(B_m/\mathcal{I}_m) = 0$ and that

$$\mathbf{K_0}(B_m/\mathcal{I}_m) = \mathbb{Z}_{|\chi(\Gamma')|}p'_{m+1*}([P_m]_0) \oplus \mathbb{Z}_{|\chi(\Gamma')|}p'_{m+1*}([P_{m-1}]_0) \oplus \cdots \oplus \mathbb{Z}_{|\chi(\Gamma')|}p'_{m+1*}([P_0]_0).$$

This completes the induction.

We have $\mathbf{K_0}(B) = \overset{\infty}{\underset{i=0}{\oplus}} \mathbb{Z}_{|\chi(\Gamma')|}[P_i]_0$ and $\mathbf{K_1}(B) = 0$. We can write $\mathbf{K_0}(B) = \overset{\infty}{\underset{i=0}{\oplus}} \mathbb{Z}_{|\chi(\Gamma')|}\beta_*^i([P_0]_0)$. Therefore $\mathbf{K_0}(\tilde{B}) = \overset{\infty}{\underset{i=-\infty}{\oplus}} \mathbb{Z}_{|\chi(\Gamma')|}\Phi_*^i([\tilde{P}_0]_0)$ and $\mathbf{K_1}(\tilde{B}) = 0$.

The Pimsner-Voiculescu exact sequence gives

$$\overset{\infty}{\underset{i=-\infty}{\oplus}} \mathbb{Z}_{|\chi(\Gamma')|}\Phi_*^i([\tilde{P}_0]_0)) \xrightarrow{\mathrm{id}_* - \Phi_*} \overset{\infty}{\underset{i=-\infty}{\oplus}} \mathbb{Z}_{|\chi(\Gamma')|}\Phi_*^i([\tilde{P}_0]_0)) \longrightarrow \mathbf{K_0}(\tilde{A})$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathbf{K_1}(\tilde{A}) \qquad \longleftarrow \qquad 0 \qquad \longleftarrow \qquad 0.$$

We conclude that $\mathbf{K_0}(\tilde{A}) = \mathbb{Z}_{|\chi(\Gamma')|}[\tilde{P}_0]_0$ and $\mathbf{K_1}(\tilde{A}) = 0$. From Proposition 2.4 we get $\mathbf{K_0}(C_Q^*(\Gamma)) = \mathbb{Z}_{|\chi(\Gamma)|}[1_{C_Q^*(\Gamma)}]_0$ and $\mathbf{K_1}(C_Q^*(\Gamma)) = 0$ (notice that $\chi(\Gamma) = \chi(\Gamma') - \chi(\Gamma_k) = \chi(\Gamma') - 0$).

From Remark 3.2 follows that $I_{\Gamma*}$ is "multiplication by $\chi(\Gamma)$", so $\mathbf{K_0}(C^*(\Gamma)) = \mathbb{Z}[1_{C^*(\Gamma)}]_0$ and $\mathbf{K_1}(C^*(\Gamma)) = 0$.

This concludes the proof of (Case **II**).

(case **III**): $\chi(\Gamma') = 0$ and $\chi(\Gamma_k) \neq 0$.

By assumption we have that $\mathbf{K_0}(B_0) = \mathbb{Z}[P_0]_0$, $\mathbf{K_0}(B_0/\mathcal{I}_0) = \mathbb{Z}p'_{1*}([P_0]_0)$, $\mathbf{K_1}(B_0) = 0$ and $\mathbf{K_1}(B_0/\mathcal{I}_0) = \mathbb{Z}$.

Suppose by induction that

$$\mathbf{K_0}(B_{m-1}) = \mathbb{Z}[P_0]_0 \oplus \mathbb{Z}_{|\chi(\Gamma_k)|}[P_1]_0 \oplus \cdots \oplus \mathbb{Z}_{|\chi(\Gamma_k)|}[P_{m-1}]_0$$

and that $\mathbf{K_1}(B_{m-1}) = 0$. Then from diagram (24) we see that the maps γ_m^{ind} and p'_{m*}: $\mathbf{K_0}(B_{m-1}) \to \mathbf{K_0}(B_{m-1}/\mathcal{I}_{m-1})$ are isomorphisms. Since also π_m is an isomorphism this implies that $I_{m*}: \mathbf{K_0}(B_{m-1}) \to \mathbf{K_0}(B_m)$ is an isomorphism into and that p_{m*} restricted to $\mathcal{G} = \langle [P_0]_0, \ldots, [P_{m-1}]_0 \rangle_{\mathbf{K_0}(B_m)}$ is also an isomorphism. From the fact that $p_{m*}|_{\mathcal{G}}$ is injective follows that there are no relations between $[P_m]_0$ and $\mathcal{G}$. The commutativity of

$$\mathbf{K_1}(B_{m-1}/\mathcal{I}_{m-1}) \xrightarrow[\cong]{\gamma_m^{\mathrm{ind}}} \mathbf{K_0}(\mathcal{I}_{m-1})$$

$$\pi_{m*} \downarrow \cong \qquad\qquad\qquad I_{m*} \downarrow [= \times \chi(\Gamma_k)]$$

$$\mathbf{K_1}(B_m/T_m) \xrightarrow{\delta_m^{\mathrm{ind}}} \mathbf{K_0}(T_m)$$

implies that δ_m^{ind} is "multiplication by $\chi(\Gamma_k)$". Thus $[P_m]_0$ in $\mathbf{K}_0(B_m)$ is of order $\chi(\Gamma_k)$ (as should be by Lemma 3.1). Therefore

$$\mathbf{K}_0(B_m) = \mathbb{Z}[P_0]_0 \oplus \mathbb{Z}_{|\chi(\Gamma_k)|}[P_1]_0 \oplus \cdots \oplus \mathbb{Z}_{|\chi(\Gamma_k)|}[P_m]_0.$$

We also showed that δ_m^{ind} is injective and therefore $\mathbf{K}_1(B_m) = 0$.

Now we easily get $\mathbf{K}_0(B) = \mathbb{Z}[P_0]_0 \oplus \overset{\infty}{\underset{i=1}{\oplus}} \mathbb{Z}_{|\chi(\Gamma_k)|} \beta_*^i([P_0]_0)$ and $\mathbf{K}_1(B) = 0$. From this follows that $\mathbf{K}_0(\tilde{B}) = \overset{\infty}{\underset{i=-\infty}{\oplus}} \mathbb{Z}_{|\chi(\Gamma_k)|} \Phi_*^i([\tilde{P}_0]_0)$ and $\mathbf{K}_1(\tilde{B}) = 0$.

The Pimsner-Voiculescu exact sequence gives

$$\underset{i=-\infty}{\overset{\infty}{\oplus}} \mathbb{Z}_{|\chi(\Gamma_k)|} \Phi_*^i([\tilde{P}_0]_0)) \xrightarrow{\mathrm{id}_* - \Phi_*} \underset{i=-\infty}{\overset{\infty}{\oplus}} \mathbb{Z}_{|\chi(\Gamma_k)|} \Phi_*^i([\tilde{P}_0]_0)) \longrightarrow \mathbf{K_0}(\tilde{A})$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathbf{K_1}(\tilde{A}) \qquad\qquad \longleftarrow \qquad\qquad 0 \qquad\qquad \longleftarrow \qquad\qquad 0.$$

We conclude that $\mathbf{K}_0(\tilde{A}) = \mathbb{Z}_{|\chi(\Gamma_k)|}[\tilde{P}_0]_0$ and $\mathbf{K}_1(\tilde{A}) = 0$. From Proposition 2.4 we get $\mathbf{K}_0(C_Q^*(\Gamma)) = \mathbb{Z}_{|\chi(\Gamma)|}[1_{C_Q^*(\Gamma)}]_0$ and $\mathbf{K}_1(C_Q^*(\Gamma)) = 0$ (notice that $\chi(\Gamma) = \chi(\Gamma') - \chi(\Gamma_k) = 0 - \chi(\Gamma_k)$).

From Remark 3.2 follows that $I_{\Gamma*}$ is "multiplication by $\chi(\Gamma)$", so $\mathbf{K}_0(C^*(\Gamma)) = \mathbb{Z}[1_{C^*(\Gamma)}]_0$ and $\mathbf{K}_1(C^*(\Gamma)) = 0$.

This concludes the proof of (Case **III**).

(cases **IV** and **V**): $\chi(\Gamma') \neq 0$, $\chi(\Gamma_k) \neq 0$.

Let's denote $x = \chi(\Gamma')$, $y = \chi(\Gamma_k)$ and let $\mathrm{GCD}(x, y) = d > 0$ be the greatest common divisor of x and y. Then by the Bézout's identity there exist $a, b \in \mathbb{Z}$ such that $ax + by = d$. Denote also $x' = x/d$ and $y' = y/d$. Then $ax' + by' = 1$.

By assumption we have that $\mathbf{K}_0(B_0) = \mathbb{Z}[P_0]_0$, $\mathbf{K}_1(B_0/\mathcal{I}_0) = 0$, $\mathbf{K}_1(B_0) = 0$ and $\mathbf{K}_0(B_0/\mathcal{I}_0) = \mathbb{Z}_{|x|} p'_{1*}([P_0]_0)$.

Then $0 = \mathbf{K}_1(B_0/\mathcal{I}_0) \cong \mathbf{K}_1(B_1/T_1)$ which implies $\mathbf{K}_1(B_1) = 0$. Diagram (24) for $m = 1$ reduces to

$$(30) \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{K}_0(\mathcal{I}_0) & \xrightarrow{i'_{1*}} & \mathbf{K}_0(B_0) & \xrightarrow{p'_{1*}} & \mathbf{K}_0(B_0/\mathcal{I}_0) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle I'_{1*}} & & \downarrow{\scriptstyle I_{1*}} & & \cong \downarrow{\scriptstyle \pi_{1*}} & & \\
0 & \longrightarrow & \mathbf{K}_0(T_1) & \xrightarrow{i_{1*}} & \mathbf{K}_0(B_1) & \xrightarrow{p_{1*}} & \mathbf{K}_0(B_1/T_1) & \longrightarrow & 0.
\end{array}$$

Clearly in $\mathbf{K}_0(B_1)$ we have $x[P_0]_0 - y[P_1]_0 = 0$. Consider $g = b[P_0]_0 + a[P_1]_0$, $g' = x'[P_0]_0 - y'[P_1]_0 \in \mathbf{K}_0(B_1)$. Since $ag' + y'g = (ax' + y'b)[P_0]_0 = [P_0]_0$ and $x'g - bg' = (x'a + by')[P_1]_0 = [P_1]_0$ it follows that g and g' generate $\mathbf{K}_0(B_1)$. Since i_{1*} is injective on $\mathbf{K}_0$ it follows that $\mathbf{K}_0(B_1)$ is an infinite group. Clearly $dg' = dx'[P_0]_0 - dy'[P_1]_0 = 0$. Therefore g is of infinite order in $\mathbf{K}_0(B_1)$ and moreover g and g' are not related (or otherwise g would be of finite order). If we suppose that $0 < d' \mid d$ and $d'g' = 0$ then it will follow that $d'x'[P_0]_0 = d'y'[P_1]_0 \in \ker(p_{1*})$. But the order of $p_{1*}([P_0]_0)$ in $\mathbf{K}_0(B_1/T_1)$ is $|x|$, so $d'x' \geq x$ or $d' \geq d$. Therefore $d' = d$ and $\mathbf{K}_0(B_1) = \mathbb{Z}(b[P_0]_0 + a[P_1]_0) \oplus \mathbb{Z}_d(x'[P_0]_0 - y'[P_1]_0)$.

Therefore we showed that $\mathbf{K}_0(B_1) = \{\mathbb{Z}[P_0]_0 \oplus \mathbb{Z}[P_1]_0 | x[P_0]_0 - y[P_1]_0 = 0\}$. Suppose by induction that for $m \geq 2$, $\mathbf{K}_1(B_{m-1}) = 0$ and that

$$\mathbf{K}_0(B_{m-1}) = \{\mathbb{Z}[P_0]_0 \oplus \cdots \oplus \mathbb{Z}[P_{m-1}]_0 | x[P_0]_0 - y[P_1]_0 = 0, \ldots, x[P_{m-2}]_0 - y[P_{m-1}]_0 = 0\}.$$

From the induction hypothesis follows that $[P_{m-1}]_0$ is of infinite order in $\mathbf{K}_0(B_{m-1})$ and therefore that $i'_{m*} : \mathbf{K}_0(\mathcal{I}_{m-1}) \to \mathbf{K}_0(B_{m-1})$ is injective and therfore $0 = \mathbf{K}_1(B_{m-1}/\mathcal{I}_{m-1}) \cong \mathbf{K}_1(B_m/T_m)$. This shows that $\mathbf{K}_1(B_m) = 0$ and that (24) reduces to

(31)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{K}_0(\mathcal{I}_{m-1}) & \xrightarrow{i'_{m*}} & \mathbf{K}_0(B_{m-1}) & \xrightarrow{p'_{m*}} & \mathbf{K}_0(B_{m-1}/\mathcal{I}_{m-1}) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle I'_{m*}} & & \downarrow{\scriptstyle I_{m*}} & & \cong\downarrow{\scriptstyle \pi_{m*}} & & \\
0 & \longrightarrow & \mathbf{K}_0(T_m) & \xrightarrow{i_{m*}} & \mathbf{K}_0(B_m) & \xrightarrow{p_{m*}} & \mathbf{K}_0(B_m/T_m) & \longrightarrow & 0.
\end{array}$$

It is easy to see that

$$\mathbf{K}_0(B_{m-1}/\mathcal{I}_{m-1}) = \{\mathbb{Z}p'_{m*}([P_{m-1}]_0) \oplus \cdots \oplus \mathbb{Z}p'_{m*}([P_0]_0) | xp'_{m*}([P_{m-1}]_0) = 0,$$
$$xp'_{m*}([P_{m-2}]_0) - yp'_{m*}([P_{m-1}]_0) = 0, \ldots, xp'_{m*}([P_0]_0) - yp'_{m*}([P_1]_0) = 0\}.$$

Since I'_{m*} is "multiplication by $\chi(\Gamma_k)$" and therefore injective then by the Five Lemma follows that I_{m*} is also injective. Therefore if we denote $\mathcal{G} = I_{m*}(\mathbf{K}_0(B_{m-1}))$ then $\mathbf{K}_0(B_m) = \langle [P_m]_0, \mathcal{G} \rangle$. One obvious relation in $\mathbf{K}_0(B_m)$ beside the relations that come from $\mathbf{K}_0(B_{m-1})$ is $x[P_{m-1}]_0 - y[P_m]_0 = 0$ and this relation follows from Lemma 3.1. Therefore $\mathbf{K}_0(B_m)$ is a quotient of the group

$$F = \{\mathbb{Z}\rho_0 \oplus \cdots \oplus \mathbb{Z}\rho_m | x\rho_{m-1} - y\rho_m = 0, \ldots, x\rho_0 - y\rho_1 = 0\},$$

where the quotient map $f : F \to \mathbf{K}_0(B_m)$ is defined on the generators as $\rho_l \mapsto [P_l]_0$, $l = 0, \ldots, m$. Then if $F' = \mathbb{Z}\rho_m$ the quotient $F_q = F/F'$ is isomorphis to

$$F_q = \{\mathbb{Z}\rho_0 \oplus \cdots \oplus \mathbb{Z}\rho_m | x\rho_{m-1} - y\rho_m = 0, \ldots, x\rho_0 - y\rho_1 = 0, \rho_m = 0\} =$$
$$= \{\mathbb{Z}\rho_0 \oplus \cdots \oplus \mathbb{Z}\rho_{m-1} | x\rho_{m-1} = 0, x\rho_{m-2} - y\rho_{m-1} = 0, \ldots, x\rho_0 - y\rho_1 = 0\}.$$

Obviously we have the commutative diagram of abelian groups with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F_q & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_q} & & \\
0 & \longrightarrow & \mathbf{K}_0(T_m) & \xrightarrow{i_{m*}} & \mathbf{K}_0(B_m) & \xrightarrow{p_{m*}} & \mathbf{K}_0(B_m/T_m) & \longrightarrow & 0,
\end{array}$$

where f_q is the homomorphism induced by f and f' is the restriction of f to F'. Then obviously f' and f_q are isomorphisms (since π_{m*} is an isomorphism). Therefore by the Five Lemma follows that f is also an isomorphism.

This shows that

$$\mathbf{K}_0(B_m) = \{\mathbb{Z}[P_0]_0 \oplus \cdots \oplus \mathbb{Z}[P_m]_0 | x[P_0]_0 - y[P_1]_0 = 0, \ldots, x[P_{m-1}]_0 - y[P_m]_0 = 0\}.$$

We also showed above that $\mathbf{K}_1(B_m) = 0$ and this concludes the induction.

Now it is easy to see that $\mathbf{K}_1(B) = 0$ and that

$$\mathbf{K}_0(B) = \{\overset{\infty}{\underset{i=0}{\oplus}} \mathbb{Z}\beta_*^i([P_0]_0) | \chi(\Gamma')\beta_*^i([P_0]_0) - \chi(\Gamma_k)\beta_*^{i+1}([P_0]_0) = 0, \ i \in \mathbb{N}_0\}.$$

Then $\mathbf{K}_1(\tilde{B}) = 0$ and

$$\mathbf{K}_0(\tilde{B}) = \{\bigoplus_{i=-\infty}^{\infty} \mathbb{Z}\Phi_*^i([\tilde{P}_0]_0) | \chi(\Gamma')\Phi_*^i([\tilde{P}_0]_0) - \chi(\Gamma_k)\Phi_*^{i+1}([\tilde{P}_0]_0) = 0, \ i \in \mathbb{Z}\}.$$

The Pimsner-Voiculescu exact sequence gives

(32)
$$\begin{array}{ccccc} \mathbf{K}_0(\tilde{B}) & \xrightarrow{\mathrm{id}_* - \Phi_*} & \mathbf{K}_0(\tilde{B}) & \longrightarrow & \mathbf{K}_0(\tilde{A}) \\ \uparrow & & & & \downarrow \\ \mathbf{K}_1(\tilde{A}) & \longleftarrow & 0 & \longleftarrow & 0. \end{array}$$

(Case **IV**): $\chi(\Gamma') \neq 0$, $\chi(\Gamma_k) \neq 0$ and $\chi(\Gamma') = \chi(\Gamma_k)$.
In this case

$$\mathbf{K}_0(\tilde{A}) = \{\bigoplus_{i=-\infty}^{\infty} \mathbb{Z}\Phi_*^i([\tilde{P}_0]_0) | \chi(\Gamma')\Phi_*^i([\tilde{P}_0]_0) - \chi(\Gamma')\Phi_*^{i+1}([\tilde{P}_0]_0) = 0,$$

$$\Phi_*^i([\tilde{P}_0]_0) - \Phi_*^{i+1}([\tilde{P}_0]_0) = 0, \ i \in \mathbb{Z}\} =$$

$$\{\bigoplus_{i=-\infty}^{\infty} \mathbb{Z}\Phi_*^i([\tilde{P}_0]_0) | \Phi_*^i([\tilde{P}_0]_0) - \Phi_*^{i+1}([\tilde{P}_0]_0) = 0, \ i \in \mathbb{Z}\} = \mathbb{Z}[\tilde{P}_0]_0.$$

To examine $\ker(\mathrm{id}_* - \Phi_*)$ take $\omega = \sum_{i=-j}^{j} t_i \Phi_*^i([\tilde{P}_0]_0) \in \ker(\mathrm{id}_* - \Phi_*)$, where $t_i \in \mathbb{Z}$.
Then

$$0 = (\mathrm{id}_* - \Phi_*)(\omega) = \sum_{i=-j}^{j} t_i(\mathrm{id}_* - \Phi_*)(\Phi_*^i([\tilde{P}_0]_0)) = \sum_{i=-j}^{j} t_i(\Phi_*^i([\tilde{P}_0]_0) - \Phi_*^{i+1}([\tilde{P}_0]_0)).$$

Therefore $t_i = s_i|\chi(\Gamma')|$ for some integers s_i, $i = -j, \ldots, j$. From this easily follows that $\omega = \sum_{i=-j}^{j} s_i|\chi(\Gamma')|[\tilde{P}_0]_0$. Thus $\ker(\mathrm{id}_* - \Phi_*) = |\chi(\Gamma')|\mathbb{Z}[\tilde{P}_0]_0$. This shows that $\mathbf{K}_1(\tilde{A}) = \mathbb{Z}$.

From Proposition 2.4 follows that $\mathbf{K}_0(\tilde{A}) \cong \mathbf{K}_0(C_Q^*(\Gamma)) = \mathbb{Z}$, $\mathbf{K}_1(\tilde{A}) \cong \mathbf{K}_1(C_Q^*(\Gamma)) = \mathbb{Z}$ and that $[1_{C_Q^*(\Gamma)}]_0$ generates $\mathbf{K}_0(C_Q^*(\Gamma))$.

From Remark 3.2 follows that in the extension (22) the map I_{Γ_*} on $\mathbf{K}_0$ is zero (since $\chi(\Gamma) = \chi(\Gamma') - \chi(\Gamma_k) = 0$). Therefore $\mathbf{K}_0(C^*(\Gamma)) = \mathbb{Z}[1_{C^*(\Gamma)}]_0$ and $\mathbf{K}_1(C^*(\Gamma)) = 0$.
This concludes the proof of (case **IV**).

(case **V**): $\chi(\Gamma') \neq 0$, $\chi(\Gamma_k) \neq 0$ and $\chi(\Gamma') \neq \chi(\Gamma_k)$.
In this case

$$\mathbf{K}_0(\tilde{A}) = \{\bigoplus_{i=-\infty}^{\infty} \mathbb{Z}\Phi_*^i([\tilde{P}_0]_0) | \chi(\Gamma')\Phi_*^i([\tilde{P}_0]_0) - \chi(\Gamma_k)\Phi_*^{i+1}([\tilde{P}_0]_0) = 0,$$

$$\Phi_*^i([\tilde{P}_0]_0) - \Phi_*^{i+1}([\tilde{P}_0]_0) = 0, \ i \in \mathbb{Z}\} =$$

$$\{\mathbb{Z}[\tilde{P}_0]_0 | \chi(\Gamma')[\tilde{P}_0]_0 - \chi(\Gamma_k)[\tilde{P}_0]_0 = 0\} = \mathbb{Z}_{|\chi(\Gamma') - \chi(\Gamma_k)|}[\tilde{P}_0]_0 = \mathbb{Z}_{|\chi(\Gamma)|}[\tilde{P}_0]_0.$$

We only need to show that $\mathbf{K}_1(\tilde{A}) = 0$ or that $\mathrm{id}_* - \Phi_*$ is injective.

Take $\omega = \sum\limits_{i=-j}^{j} t_j \Phi_*^i([\tilde{P}_0]_0)$, $t_i \in \mathbb{Z}$ and suppose that $(\mathrm{id}_* - \Phi_*)(\omega) = 0$. Then

$$0 = (\mathrm{id}_* - \Phi_*)(\omega) = \sum\limits_{i=-j}^{j} t_j (\Phi_*^i([\tilde{P}_0]_0) - \Phi_*^{i+1}([\tilde{P}_0]_0)) =$$

$$= t_{-j}\Phi_*^{-j}([\tilde{P}_0]_0) + \sum\limits_{i=-j+1}^{j} (t_i - t_{i-1})\Phi_*^i([\tilde{P}_0]_0) - t_j\Phi_*^{j+1}([\tilde{P}_0]_0).$$

If $\chi(\Gamma')$ doesn't divide t_{-j} then the equality $-t_{-j}\Phi_*^{-j}([\tilde{P}_0]_0) = (t_i - t_{i-1})\Phi_*^i([\tilde{P}_0]_0) - t_j\Phi_*^{j+1}([\tilde{P}_0]_0)$ is impossible. If $\chi(\Gamma')$ divides t_{-j} then ω can be expressed in terms of $\Phi_*^{-j+1}([\tilde{P}_0]_0), \ldots, \Phi_*^j([\tilde{P}_0]_0)$. By induction we see that we can write $\omega = t[\tilde{P}_0]_0$ for some $t \in \mathbb{Z}$. But then clearly $(\mathrm{id}_* - \Phi_*)(\omega) = 0$ is possible if and only if $t = 0$. This shows that $\mathrm{id}_* - \Phi_*$ is injective and therefore that $\mathbf{K}_1(\tilde{A}) = 0$.

From Proposition 2.4 we get $\mathbf{K}_0(C_Q^*(\Gamma)) = \mathbb{Z}_{|\chi(\Gamma)|}[1_{C_Q^*(\Gamma)}]_0$ and $\mathbf{K}_1(C_Q^*(\Gamma)) = 0$.

From Remark 3.2 follows that $I_{\Gamma*}$ is "multiplication by $\chi(\Gamma)$", so $\mathbf{K}_0(C^*(\Gamma)) = \mathbb{Z}[1_{C^*(\Gamma)}]_0$ and $\mathbf{K}_1(C^*(\Gamma)) = 0$.

This concludes the proof of (Case **V**).

The Proposition is proved. $\qquad\square$

Now we can apply the Kirchberg-Phillips Classification theorem ([16]) to $C_Q^*(G)$ for a finite graph G such that G^{opp} is connected and with at least two vertices, using Theorem 1.1, Proposition 2.4 and Proposition 3.3. We obtain

$$(33) \qquad\qquad C_Q^*(G) \cong \mathcal{O}_{1+|\chi(G)|}.$$

For infinite graphs with connected opposite graphs we can argue similarly as in [8, Corollary 3.11] to prove the following:

Proposition 3.4. *Let G an infinite graph with countably many vertices and such that G^{opp} is connected. Then $C^*(G)$ $(= C_Q^*(G))$ is nuclear and belongs to the small bootstrap class. Moreover $\mathbf{K}_0(C^*(G)) = \mathbb{Z}[1_{C^*(G)}]_0$ and $\mathbf{K}_1(C^*(G)) = 0$.*

Proof. By induction we will find a increasing sequence G_n of subgraphs of G with n vertices, $n \geq 2$ which are such that G_n^{opp} is connected for each $n \geq 2$ and also $G_n \overset{n\to\infty}{\longrightarrow} G$. Obviously we can find two vertices v_1 and v_2 that are not connected (since G^{opp} is conected). Then we chose G_2 to be the graph with vertices v_1 and v_2 and no edges. Suppose we have defined the subgraph G_n for some $n \geq 2$. Let $v_1, \ldots, v_n$ be the vertices of G_n. Since G^{opp} is connected we can find a vertex v_{n+1} of G different from $v_1, \ldots, v_n$ such that v_{n+1} is not connected with all of the vertices $v_1, \ldots, v_n$. Then obviously the subgraph G_{n+1} of G on vertices $v_1, \ldots, v_{n+1}$ and edges comming from G is such that G_{n+1}^{opp} is connected. This completes the induction.

From Proposition 3.3 we have $\mathbf{K}_0(C^*(G_n)) = \mathbb{Z}[1_{C^*(G_n)}]_0$ and $\mathbf{K}_1(C^*(G_n)) = 0$. It is easy to see that $C^*(G) = \varinjlim C^*(G_n)$. Therefore from Proposition 2.4 we get that $C_Q^*(G)$ is nuclear and belongs to the small bootstrap category $\mathfrak{N}$. Also $\mathbf{K}_0(C^*(G)) = \lim\limits_{n\to\infty} \mathbf{K}_0(C^*(G_n)) = \mathbb{Z}[1_{C^*(G)}]_0$ and $\mathbf{K}_1(C^*(G)) = \lim\limits_{n\to\infty} \mathbf{K}_0(C^*(G_n)) = 0$.

This proves the proposition. $\qquad\square$

From Theorem 1.1 we know that $C^*(G) = C^*_Q(G)$ is purely infinite and simple. Again using Kirchberg-Phillips theorem we get that if G is an infinite graph on countably many vertices such that G^{opp} is connected then $C^*(G) = C^*_Q(G) \cong \mathcal{O}_\infty$. If we define for an infinite countable graph G with G^{opp} conected $\chi(G) \overset{def}{=} \infty$ then we can write once again $C^*_Q(G) \cong \mathcal{O}_{1+|\chi(G)|}$.

Remark 3.5. *Let G_1 and G_2 be two disjoint graphs. Then by $G_1 * G_2$ we denote their join which is the graph obtained from G_1 and G_2 by connecting each vertex of G_1 with each vertex of G_2. Then if we start with a graph G on countably many vertices which is such that G^{opp} doesn't have any isolated vertices then we can find a sequence of subraphs G_n, $n \in \mathbb{N}$ (some of G_n's can have zero vertices) such that G_n^{opp} are all connected and such that $G = \overset{\infty}{\underset{n=1}{*}} G_n$. For a graph F with zero vertices we write $C^*_Q(F) = \mathbb{C}$.*

Then from Theorem 1.1 easily follows that $C^*_Q(G) = \overset{\infty}{\underset{n=1}{\otimes}} C^*_Q(G_n)$.

Now we can record our main result:

Theorem 3.6. *Let G be a graph with at least two and at most countably many vertices such that G^{opp} has no isolated vertices. Write $G = \overset{\infty}{\underset{n=1}{*}} G_n$ as in Remark 3.5 with G_n being a subgraph of G such that G_n^{opp} is connected.*
Then

$$\text{(34)} \qquad C^*_Q(G) = \overset{\infty}{\underset{n=1}{\otimes}} C^*_Q(G_n) \cong \overset{\infty}{\underset{n=1}{\otimes}} \mathcal{O}_{1+|\chi(G_n)|}.$$

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